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Challenging the School Mathematics Culture:

An Investigative Small-Group Approach

Ethnographic Teacher Research on Social and Sociomathematical Norms

Academic Dissertation to be publicly defended
with permission of the Faculty of Education at the University of Lapland,
in the Esko ja Asko Hall on Saturday 28th of May 2011 at 12 o'clock.

University of Lapland
Faculty of Education

Distribution: Lapland University Press
P.o. Box 8123
FI-96101 Rovaniemi

tel. + 358 (0)40 821 4242 , fax + 358 16 362 932
publication@ulapland.fi
www.ulapland.fi/lup

Paperback
ISBN 978-952-484-463-5
ISSN 0788-7604

pdf
ISBN 978-952-484-495-6
ISSN 1796-6310
www.ulapland.fi/unipub/actanet

ABSTRACT

Partanen Anna-Maija

Challenging the School Mathematics Culture: An Investigative Small-Group Approach

In the autumn of 2001 I conducted a teaching experiment with my second year students (aged 16 – 17 years). The experiment was part of my PhD project and it took place in the Finnish upper secondary school Lyseonpuiston lukio. Instead of teaching the basic concepts of calculus in a traditional way, I gave my students questions and problems to be solved in small-groups. My intention was that, while investigating and discussing mathematics, the students would construct important aspects of the concepts of limit and derivative by themselves. The students were allowed to choose their small-group companions and, with few exceptions, this resulted in either all-female or all-male groups.

During the experimental course and while transcribing the discussions of the small-groups, I noticed that the new approach did not fit neatly into the school mathematics teaching tradition. There was friction of certain type preventing us, including myself, from acting in ways compatible with the new approach. I also noticed that the interaction was different in the girl and boy groups. I chose the general ethnographic approach for my research. I wanted to understand and describe the important aspects of interaction and learning taking place in the small groups of my experiment.

The emergent perspective developed by Erna Yackel and Paul Cobb (1996, 1998) appeared relevant in trying to make sense of what went on in my classroom and small-groups. I decided to apply the framework to my naturalistic study, in which an ordinary mathematics teacher attempted to develop her practice by adopting a new instructional approach. The aim of the research is to describe the ecology of social and sociomathematical norms in my experimental classroom by analyzing the norms that were negotiated and produced in the peer and teacher-students interactions of two small-groups. An additional aim is to shed light on how acting according to the norms was intertwined with the occurrence of learning opportunities for the students. My primary data consists of video recordings of six sessions by two small-groups (group A: two girls and a boy and group B: four boys). The additional data includes pre- and post tests, learning diaries of the students, and my own diary. In my analysis I applied the method developed by Paul Cobb (1995). It is analogous to microethnographic analysis of interaction in terms of its general view of data. In line with the ethnographic approach, I have also used my acquaintance with the school and the students when constructing interpretations of the occurrences in the data. My project is part of the “teachers as researchers” tradition.

My results show that new norms should be established for the investigative small-group approach to realize its full potential. By giving investigations to my students and by prompting them to construct and express their own ideas, I negotiated that it is the role of the students to express their own thinking and create new mathematics. However, my negotiation was not consistent. Occasionally, I acted according to the belief that it is the role of the students to find the *official* meanings of mathematical objects. The students tried to fulfill my new expectations, but they also often acted according conventional norms by just following the instructions given by the teacher, even at previous lessons, or by trying to apply ready made knowledge. The investigative approach requires that, instead of solving tasks quickly at the superficial level, students should approach mathematical problems in a profound and creative way. For the new approach, we also negotiated the different methods norm: in addition to the symbolic method, also numerical and graphical methods, as well as drawing and writing, are approved when investigating mathematics. Acting according to these new norms contributed to the occurrence of learning opportunities.

On the basis of the results, it is clear that in our face-to-face interactions we were not accustomed to justify our mathematical arguments. Rather, statements were generally accepted for social reasons, that is, on the basis of authority or social agreement. This destroyed many learning opportunities. However, in spite of the lack of justifying, the students seemed to have the sound belief that an explicit justification in a mathematical discussion must be based on the properties of the mathematical objects at hand.

There were differences between the two small-groups in their style of interaction. In group B the students justified their arguments more often and challenged one another by disagreeing verbally. In group A, ostentatious disagreeing was difficult for the students, and they indicated agreement much more often than the other group. Participation was not democratic in the small-group discussions. Those students who were assertive had the right to express their ideas, while the others had the obligation to listen. Thus, many learning opportunities were lost. These features of our interactions may reflect the different sociolinguistic subcultures of girls and boys. On the basis of my findings I criticize the emergent perspective of focusing solely on the classroom level. Sometimes it may be necessary to take into account other cultural processes coming from outside the school environment in order to make sense of the events taking place in the classroom. In this respect, I don't find the emergent perspective and the elaborations carried out so far completely satisfactory.

For further developmental work, I have constructed a scheme for negotiating norms more compatible with the investigative small-group approach.

Key words: emergent perspective, social norms, sociomathematical norms, learning opportunities, small groups in instruction, gender and language, teacher research, school ethnography

TIIVISTELMÄ

Partanen Anna-Maija

Tutkiva pienryhmätyöskentely haastaa koulumatematiikan kulttuurin

Syksyllä 2001 suoritin väitöstutkimukseeni liittyvän opetuskokeilun koulussani Lyseonpuiston lukiossa Rovaniemellä. Lukion toisen vuosikurssin opiskelijoiden (16 – 17 vuotta) kanssa opiskelimme differentiaalilaskennan peruskäsitteet pienryhmissä tutkien. Sen sijaan, että olisin opettanut kurssin perinteisellä tavalla, annoin opiskelijoille kysymyksiä ja ongelmia ratkaistavaksi yhdessä toisten kanssa. Tavoitteena oli, että tutkimisen ja keskustelun kautta opiskelijat, jos mahdollista, rakentavat itselleen raja-arvon ja derivaatan käsitteisiin liittyviä tärkeitä merkityksiä. Opiskelijat saivat itse muodostaa pienryhmät, ja muutamaa poikkeusta lukuun ottamatta kaikki olivat joko tyttö- tai poikaryhmiä.

Kokeilukurssin aikana ja kirjoittaessani opiskelijoiden puhetta muistiin videonauhalta, huomasin, että uusi työmuoto ei niin vain istunutkaan koulumatematiikan perinteisiin. Koin tiettyä ”kitkaa”, joka esti minua ja opiskelijoita toimimasta johdonmukaisesti uuden lähestymistavan kanssa. Huomasin myös, että tyttö- ja poikaryhmien vuorovaikutus oli jollain tavalla erilaista. Valitsin tutkimukselleni etnografisen lähestymistavan. Halusin ymmärtää ja kuvata pienryhmien vuorovaikutusta sekä siinä tapahtuvaa oppimista. Tutkimukseni liittyy ”opettaja oman työnsä tutkijana”-perinteeseen.

Erna Yackelin ja Paul Cobbin (1996) kehittämä emergentti viitekehys tuntui kuvaavan osuvasti niitä ilmiöitä, joita olin havainnut kokeilukurssilla ja pienryhmissä. Päätin soveltaa teoriaa omaan tilanteeseeni, jossa tavallinen lukion matematiikan opettaja haluaa kehittää omaa työtään kokeilemalla uutta opetusmenetelmää. Tutkimukseni tavoite on kuvata mahdollisimman monipuolisesti kokeilukurssilla ilmenneitä sosiaalisia ja sosiomatemaattisia normeja. Tähän tavoitteeseen pyrin analysoimalla, millaisia normeja neuvoteltiin ja tuotettiin kahden pienryhmän vuorovaikutuksessa. Lisäksi tavoitteena on valaista sitä, miten normien mukainen toiminta on yhteydessä oppimismahdollisuuksien syntymiseen. Ensisijainen aineistoni koostuu videonauhoituksista, jotka on taltioitu kahden pienryhmän (ryhmä A: kaksi tyttöä ja poika ja ryhmä: B neljä poikaa) työskentelystä kuuden tapaamiskerran aikana. Sen lisäksi minulla oli käytettävissäni opiskelijoiden vastaukset lähtötasotestissä ja loppukokeessa, opiskelijoiden oppimispäiväkirjat sekä omat päiväkirjani kokeilukurssin ajalta. Analyysissani sovelsin Paul Cobbin (1995) kehittämää menetelmää, jolla on tiettyjä yhtäläisyyksiä mikroetnografisen vuorovaikutusanalyysin kanssa. Etnografisen tutkimusotteen mukaisesti olen aineiston ilmiöiden tulkinnassa hyödyntänyt sitä, että tunsin hyvin koulun ja kokeilukurssin oppilaat.

Tulokseni osoittavat, että jotta saisimme hyödynnettyä tutkivan pienryhmätyöskentelyn suomat mahdollisuudet matematiikan opiskelussa, sen

käyttöönotto vaatii uudenlaisten sosiaalisten ja sosiomatemaattisten normien neuvottelua. Kun annoin opiskelijoilleni tutkimustehtäviä ja kehotin heitä muodostamaan omia ideoita ja keskustelemaan niistä, neuvottelin samalla sosiaalista normia, jonka mukaan tutkivassa työskentelyssä opiskelijoiden rooli on ilmaista omaa ajatteluaan ja luoda uutta matematiikkaa. Neuvotteluni opiskelijoiden kanssa ei kuitenkaan ollut johdonmukaista. Toisinaan käyttäytymiseni viesti uskomuksesta, että opiskelijoiden tehtävä on löytää matemaattisten objektien *oikeat* ja viralliset merkitykset. Opiskelijat pyrkivät toimimaan uusien odotusteni mukaisesti, mutta usein heilläkin oli siinä vaikeuksia. Joskus he toteuttivat roolia, jonka mukaan opiskelijoiden tehtävä on seurata opettajan antamia ohjeita, vaikkapa edelliseltä tunnilta. Joskus he etsivät valmista tietoa, jota voisivat soveltaa. Onnistuakseen tutkiva työskentely vaatii, että nopeiden ja pintapuolisten ratkaisuyritysten sijaan opiskelijat paneutuvat ongelmiin syvällisesti ja luovasti. Uutta työmuotoa varten neuvottelimme myös normia, jonka mukaan numeeriset ja graafiset menetelmät sekä piirtäminen ja kirjoittaminen ovat hyväksyttäviä ratkaisumenetelmiä symbolisten menetelmien ohella. Näiden uusien normien mukaan toimiminen edisti oppimismahdollisuuksien syntymistä.

Minä ja oppilaani emme olleet tottuneet perustelemaan matemaattisia väitteitämme ainakaan lähivuorovaikutuksessamme. Sen sijaan väitteitä hyväksyttiin tai hylättiin sosiaalisin perustein, joko auktoriteettiin luottaen tai yksimielisyyteen nojautuen. Useita oppimismahdollisuuksia menetettiin tästä syystä. Vaikka matemaattisia väitteitä perusteltiin harvoin, niin tulokseni kuitenkin viittaavat opiskelijoiden uskomukseen, että pätevän matemaattisen perustelun tulee nojautua matemaattisten objektien ominaisuuksiin.

Kaksi tutkittua pienryhmää erosivat toisistaan vuorovaikutustyyliltään. Ryhmässä B opiskelijat perustelivat väitteitään useammin kuin toisessa ryhmässä, ja he myös haastoivat toisiaan ilmaisemalla erimielisyyksiä. Ryhmän A jäsenille eri mieltä oleminen tuntui olevan vaikeaa. Keskusteluissaan he ilmaisivatkin yksimielisyyttä paljon enemmän kuin toisessa ryhmässä. Oikeus osallistua pienryhmien keskusteluun ei jakautunut demokraattisesti. Ne opiskelijat, jotka toivat itseään esille enemmän kuin muut, saivat puhua, ja muiden tehtävänä oli kuunnella. Monia oppimismahdollisuuksia tuhoutui tästä syystä. Edellä kuvatut vuorovaikutuksen piirteet voivat heijastaa tyttöjen ja poikien erilaisia sosiolingvistisiä alakulttuureja. Tulosteni perusteella esitän kritiikkiä emergenttiä viitekehystä kohtaan. Sen käsitteet liittyvät pelkästään luokkahuoneen mikrokulttuuriin. Mutta joskus luokkayhteisön vuorovaikutuksen ymmärtämisessä on tarpeellista huomioida myös koulun ulkopuolelta tulevat kulttuuriset vaikutteet. En ole täysin tyytyväinen viitekehyksen tämänhetkisiin laajennuksiin.

Jatkossa tapahtuvaa kehitystyötä silmällä pitäen olen rakentanut kehyksen, jonka mukaan tutkivaa pienryhmätyöskentelyä tukevien normien neuvottelu voidaan aloittaa.

Hakusanat: emergentti viitekehys, sosiaaliset normit, sosiomatemaattiset normit, oppimismahdollisuus, pienryhmä opetuksessa, sukupuoli ja kieli, opettajatutkimus, kouluetnografia

ACKNOLEDGEMENTS

The students in the original four small groups and in the final two groups made this study possible. I am grateful to all of them for letting me to video record their discussions and interactions. I would also like to express my thanks to our former Head Hely Kalkkinen and to my colleague Aila Naasko for helping in the special arrangements that were needed due to my experimental course and data collection.

Professor John Berry and Head teacher Roger Fentem, the University of Plymouth, suggested a two-year development and research project for writing and testing investigative teaching materials. Collaboration with them was a great source of inspiration. During the years, I have also enjoyed many discussions with different teachers who were enthusiastic in developing their practice and thus gave me strength to continue with my project: Raija Arajärvi, Eeva Heikkilä, Raimo Koponen, Mikko Lumme, Risto Matveinen, Marjatta Rautiainen, Tuula Saraniemi and Tuija Ylipeura, for example. Thanks to all of them.

Professor Raimo Rajala and Docent Raimo Kaasila were my main supervisors. In addition, Head teacher Veikko Keränen, Rovaniemi University of Applied Sciences, helped me in planning the studies of mathematics that were part of my study plan. I want to give special thanks to Raimo Kaasila who patiently read and commented my numerous texts and this way taught me scientific writing. I am happy that my supervisors gave me the freedom of choosing my own way of doing research. I want to express my thanks to my reviewer and opponent-to-be Professor Markku Hannula. During my studies he has encouraged me many times by small acts but exactly at right times. I am profoundly grateful to Professor Koeno Grvemeijer for our discussions, his comments on my 90 % seminar and his reviewing of my thesis. It was an honor to enjoy his expertise.

The Finnish Graduate School of Mathematics, Physics and Chemistry Education made it possible to start doctoral studies in mathematics education with my background of Master's degree in mathematics. I also had the privilege of working full-time for two years as a research student in the research school. These possibilities were indispensable for starting and finishing my project. In the research school I learnt to know Ann-Sofi Röj-Lindberg and Heidi Krzywacki. I want to thank both of them for all the long, wonderful and interesting discussions we had about theories in education and doing research. I would like to thank the Nordic Graduate School of Mathematics Education of their courses of very high quality. And in the University of Lapland, the seminar group of Gender Studies, led by Professor Päivi Naskali, and the courses by the Department of Research Methodology taught by Jukka Mäkelä and Suvi Ronkainen have broadened my perspectives about knowledge and doing research.

I am very grateful to Laskentaväline Oy which presented our school with class full of symbolic calculators TI – 92. Investigating the concepts of calculus by using different methods was highly supported by the technology available. The prize given to me by the Centennial Foundation of Technology Industry of Finland was a great encouragement and helped me to participate in some courses of the Nordic Graduate School in Norway. Thanks also to the City of Rovaniemi for partly funding my participation at the 26th Conference of the International Group for the Psychology of Mathematics Education in Norwich, UK. Special thanks to my father Mikko Suomalainen who presented me with two laptops during my studies.

Thanks to all the persons who proofread my texts written in English: Ros Cooper, Julie-Ann Edwards, Eija Koivuranta, Annukka Kurvinen and Aimo Tattari as well as to Tuula Saraniemi, who did the proofreading of my pages written in Finnish.

To my husband Pekka Partanen, I am profoundly grateful. He has shown progressive attitudes towards gender roles in family and society by letting me fulfill my dreams. The only thing I might regret after the process is that when my daughter and son were schoolchildren they had a very busy mother. However, they both have encouraged me in several ways during the years.

Rovaniemi, May 2011

Anna-Maija Partanen

CONTENTS

1 INTRODUCTION

1.1 Teaching and learning calculus in school mathematics tradition

My first teaching experiments when I asked my upper secondary students to investigate mathematics made me upset and gave me a lot to think about. I remember a lesson where I asked the students to find some values of the function $f(x) = e^x$ and evaluate the gradients of tangents at the same x-values. After the investigation a student, who later did very well in the national exams, wrote a conclusion that the derivative function of e^x is xe^{x-1}. Although there was simple and concrete evidence for the construction of the conjecture that the derivative of e^x is the function itself, the student refused to think by herself and used a rule which was already taught to her and seemed to be the closest to the situation. While having more opportunities for understanding my students' thinking, I started, little by little, doubting the quality of my teaching. Is mathematics for my students just a game for which you learn the rules but which has no meaning to them? Calculus was the area where I saw the greatest problems.

I am not, however, alone in my observations. Many researchers in mathematics education report similar situations (Cobb and Yackel, 1998, Nunes, Scliemann and Carraher, 1993, Sfard and Linchevski, 1994, Thompson, 1994, Walkerdine, 1988). Students can be effective in lessons and even judged as mathematically competent if they only show mastering of rules regardless of the meaning of the rules to them (Cobb and Yackel, 1998, Schoenfeld, 1987). The mathematical practices in many classrooms can be characterized as procedural instructions. This means that classroom discussions do not require that manipulating of symbols has the meaning of acting mentally on taken-as-shared mathematical objects (Cobb and Yackel, 1998.) Cobb and Yackel (1998) follow Richards (1991) in calling such tradition established in mathematics classrooms as *school mathematics tradition*.

Emphasis of rules in mathematics instruction, instead of trying to promote students' construction of conceptual knowledge, is especially problematic in calculus. In this field students meet several new and difficult concepts: limits, continuity of a function, derivative and integrals (Artigue, 1991, Cornu, 1991, Haapasalo et al., 1995, 1997, Lehtinen et al., 1997, Merenluoto, 2001, Tall and Vinner, 1981). Many research reports (Orton 1983a, 1983b, Baker, Cooley and Trigueros, 2000) show that students may perform well in traditional exams in calculus but at the same time have great difficulties in understanding the underlying conceptual structures. In Finland Haapasalo et al. (1995, 1997) and Merenluoto (2001) arrived in their research projects at the conclusion that Finnish upper secondary students' mastery of the concepts of limit and continuity of a function is poor. Textbooks focus on developing skills for solving mechanical tasks (Haapasalo et al., 1995, 1997). According to Repo (1996) a characteristic of teaching calculus in Finland is an emphasis on algorithms. Students learn to solve typical tasks by pre given algebraic methods. In her quasi experimental research Repo (1996) found that from the group experiencing traditional mathematics teaching, only a few of the high achievers constructed a hierarchical knowledge structure of the concept of derivative. Viholainen (2008) found that the prospective mathematics teachers in his study had difficulties in combining informal and formal reasoning about the concept of derivative in problem-solving situations.

1.2 New approaches in mathematics education

Several learning theoretical frameworks and new methods for teaching mathematics were around when I searched my way to improve my instruction and construct a teaching experiment for my thesis. One of them was constructivism. To be precise, constructivism is an epistemological and psychological theory (Cobb, 1994, von Gasersfeld, 1983) and it should not be treated as an axiomatic foundation for deducing pedagogical principles. However, it can act as a general orienting framework in which to address pedagogical issues and develop instructional approaches (Cobb, 1994.) Constructivism means a shift of focus in teaching, as well as in research, closer to the student. What matters is what happens in the minds of the students, how actively they construct knowledge and what is the quality of their constructions. Von Glasersfeld (1995b) emphasizes that radical constructivism discusses the construction of conceptual knowledge. He says his interest is in performance only insofar as it springs from understanding. According to Davis et al. (1990) mathematics teaching based on constructivism must take account of the existing knowledge of students and it must emphasize structures in knowledge. Repo states (1996) that the task of instruction is to organize learning environments where students can, according to their own level, actively construct their knowledge structures.

Constructivism has inspired teachers and researchers in developing student centered working methods for mathematics instruction. In 1980's the use of different types of open problems spread all over the world in mathematics classrooms and research on their possibilities was intense (Pehkonen, 1997). The Japanese had developed a method of using open-ended problems for teaching mathematics and promoting discussion in the classrooms (Nohda, 2000). In Great Britain the ideals of *investigation* were put forward for example by Ernest (1991), Lerman, (1989) and Morgan (1996). Among mathematics educators in Britain, investigations have become associated, as an opposition to practicing skills and reproducing standard solutions, to methods and tasks that are exploratory, open, creative and empowering. Investigations normally have multiple solutions. In the hands of curriculum and test designers and teachers, however, the idea of investigation has been stereotyped to refer to a standard algorithm of generating data through examples, organizing the data and searching for patterns in it (Morgan, 1997.) Other types of open problems used in different classrooms around the world are real-life situations, projects, problem posing, problem fields or sequences, problems without a question and problem variations (Pehkonen, 1997). In Finland Pehkonen (Pehkonen and Zimmerman, 1990, Pehkonen, 1991, 1997, 2000), Haapasalo (1994, 2004) and Leppäaho (2007) have specialized in research on mathematical problem solving in instruction.

Simultaneously with the boom in problem solving an interesting branch of research and teaching experiments in calculus was established. Before teaching algorithms, students were given possibilities for constructing the basic concepts of the field during a period of working in a CAS (computer algebra system) environment. The students were given tasks or problems to be solved, and most often the use of different representations (symbolic, graphical and numerical) was emphasized. Heid (1988) was among the first researchers who revised the traditional order of teaching skills before concepts. According to Tall (1985, 1986) students should first experience a global and qualitative introduction to a mathematical concept. This introduction, then, should create the need for a more formal description or definition. Repo (1996) and Asiala et al. (1997) emphasize the need for concept analysis, or genetic decomposition, of the concept of derivative before constructing the critical activities for the students in the CAS environment. Berry and Nyman (2003) show an example problem and argue that before teaching formal symbolic calculus students should be given opportunities of constructing the underlying concepts by working with problems like theirs. Artigue (2005) introduced the definition of the derivative by letting students conjecture and test slopes and equations of tangents by a symbolic calculator (equivalent to a CAS computer program).

During the last decades or so there has been a growing interest in the relationship between social processes and individual learning among the researchers in mathematics education. A great amount of empirical data suggests that social interaction may have a strong impact on the contents and processes of learning (Waschescio, 1998). This has led researchers in the constructivist tradition to revise their emphasis on solely the constructive activity of the individual, more or less independent of social and cultural influences. Two branches of social constructivism emerged. Some, mainly American researchers (Brown and Campione, 1994, Brown, Collins and Duguid, 1989, Greeno, 1998 and Pea, 1994) discussed the concepts of socio-cultural theories in local interactional settings. Another branch coordinates perspectives of radical constructivism, symbolic interactionism and ethnomethodlogy (Cobb and Yackel, 1996, 1998). On the other hand, a great number of researchers working in the socio-cultural (or cultural-historical) tradition have conceptualized the social aspects of development and enculturation in varied ways (for example Lave and Wenger, 1991, Rogoff, 1994, 2003, Wells, 1999, Wenger, 1998).

There is no agreement among the researchers in mathematics education about the distinction between the first type of social constructivism and socio-cultural theorizing. After 1970's the ideas of Vygotsky and soviet activity theorists have inspired researchers in psychology and education in the West. Some researchers have rather loosely applied their concepts and ideas, also in mathematics education. Others, for example Chaiklin (2003) and Lerman (1996), call for stricter adherence to the original ideas and their contexts. According to Chaiklin (2003) the emphasis of Vygotsky is on the development of children in cultural and historical situations. The branch of research following Vygotsky that applies his concepts in locally social settings is sometimes classified as social constructivism and sometimes as socio-cultural theorizing.

For social constructivists the local interactions of people when they negotiate meaning is the context where individuals construct their own understandings. Along the interactions, a taken-as-shared basis for communication is established. But according to Gravemeijer (1997) social constructivism, as such, does not offer heuristics for developing a teaching approach compatible with constructivist epistemology and therefore a supporting instructional theory is needed. In *realistic mathematics education* (Treffers, 1987, Gravemeijer, 1997, Gravemeijer and Doorman, 1999) situated and informal knowledge and strategies of students are taken as a starting point. A learning sequence consists of carefully selected contextual problems that give rise to a variety of solution methods. The solutions are discussed, compared, justified and their efficiency is evaluated. The aim is to teach formal mathematical knowledge, for example the procedure of long division or the concept of derivative, by letting it evolve from the informal knowledge and strategies of students. The role of the teacher is to choose the instructional activities, to begin and guide discussions and to reformulate selected aspects of students' mathematical activity (Gravemeijer, 1997.) For example, Cobb et al. (Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997, Cobb, 1999, Cobb, 2002, Gravemeijer, Cobb, Bowers and Whitenack, 2000) have applied realistic mathematics education in their numerous teaching experiments. They call the approach and corresponding culture as *inquiry mathematics tradition* as an opposition to school mathematics tradition.

Compatible with the student centered ideas of constructivism is the use of small-group discussions in teaching (Bennett et al., 2010, Good, Mulryan and McCaslin, 1992). Social constructivism and socio-cultural theorizing have further highlighted the importance of possibilities for students to construct, share and elaborate together their own ideas and thinking. The latter two traditions also offer theoretical frameworks for analyzing and developing methods of analysis for situated construction of knowledge and development of high-order skills in the interactions of small-groups. For example Cobb (1995) analyzed the development of second graders' conceptions of place-value numeration and their construction of increasingly sophisticated computational algorithms in connection to the ways of interacting in pairs of pupils. In Finland Kaartinen and Kumpulainen (Kaartinen and Kumpulainen, 2001, 2002, Kumpulainen and Kaartinen, 2000, 2003) have studied the cognitive and interactional processes in small-groups in science and mathematics classrooms from a sociocultural perspective and in Norway Bjuland (2002, 2007) has investigated the characteristics of mathematically productive discourses when student teachers collaboratively solve geometrical problems. A general result from the research on the use of small-groups during the last decades is that the approach as such does not promote students' learning. The use of small-groups needs to be connected to "good instruction" in which case there is evidence of better conceptual understanding and development of high order skills (Bennett et al., 2010, Good, Mulryan and McCaslin, 1992.)

1.3 How I ended up at my project

I remember a workshop organized for in-service secondary teachers in the spring of 1995. Professor John Berry from the University of Plymouth was introducing to us the ideas of investigations. Suddenly he posed a question: Can we use investigations to teach new mathematics to the students. That was a critical question made at critical moment in my career as a teacher. The idea has fascinated me ever since. I first did some small experiments by myself but in the end of the 1990's I was offered a possibility to participate in a project led by Berry and Roger Fentem from the College of St Mark and St John in Plymouth. Together with Sirkka Tiihala from Keminmaan lukio we developed and tested materials for technology supported investigative approaches in two high level courses of the contemporary Finnish upper secondary mathematics syllabus; Functions and equations 2 and Differential calculus 1 (Opetushallitus, 1994). In the investigations we used TI-92 symbolic calculators, which basically have the CAS-program Derive built in them. Some results from our project were also published (Berry, Fentem, Partanen and Tiihala, 2004).

Participating in the developmental and research project inducted me into the world of research. I noticed that there are systematic ways of making sense of the processes in my classrooms and that it is possible to try to further develop my teaching practice on the basis of those understandings. I decided to make up a research project closely connected to my own interests of developing the use of the investigative approach in my teaching. My study is part of the "teachers as researchers" tradition (Stenhouse,1975, Kincheloe, 1991, Cochran-Smith and Lytle, 1999, Altrichter et al., 2008), and enhancing my own practice is equally important as producing new knowledge in the field.

My experiences with the investigative approach made me wonder how much guidance or freedom students should be given when they investigate mathematics. On one hand my aim was to teach mathematics according to the syllabus and on the other hand I realized that the new approach emphasizes students' own sense making processes. Is it beneficial or even possible to force students' thinking processes to certain paths? My original research question was about the openness of the investigative approach. My intention was to construct a quasi experimental study about two different variations of the investigative approach and to compare them. In the teaching experiment of my PhD project students in two of my courses studied the basics of calculus utilizing the TI-92 calculators. One class used investigations which I had developed in our project, and to the other class I gave as open questions as I dared to let them work with about the same topics. In both of the classes during the investigations the students worked in friendship small-groups of three to four students.

At the time of the teaching experiment of my thesis I was to some extent aware of the constructivist ideas about learning. I had participated in a course or two led by Professor Berry. In our project I had written materials for what I thought might be an investigative approach in an upper secondary school. When I look back at my materials now, I see they give quite much guidance for the students. Yes, everybody was working with the materials and had to think by themselves, not only the brightest students. But the guidelines of the investigations followed my traditional way of teaching the subject matter. I had read about Repo’s (1996) research on teaching the concepts of calculus by a CAS-program. But on the basis of the research report it was hard to imagine what the actual process of studying was like. I consider me at that time a normal upper secondary mathematics teacher who had a little in-service training in the use of the investigative approach and symbolic calculators and who was interested, but not specialized, in the contemporary learning theoretical ideas.

My studies on research methodology in social and educational sciences and reflecting my experiences from our project, however, led me to change my plans. I gave up the idea of an experiment because it is a research methodology typical of the positivist or post-positivist paradigms (Carr and Kemmis, 1986, Kincheloe, 1991, Heikkinen, Huttunen, Niglas and Tynjälä, 2005), and there are several reasons why my project does not fit with that tradition. I was the teacher and the researcher at the same time. The development of my own practice and my personal growth as a teacher were important goals in my research. Thus, first and foremost, the positivist requirement of objectivity could not be satisfied. And, with Pring (2000), I agree that only teachers can fully understand the complexity of phenomena in an educational setting. I think that one can never control all the relevant variables at a sufficient degree to construct a proper experiment at school.

At the same time when rejecting the idea of an experiment I decided to focus on the learning processes instead of the learning outcomes. That was recommended by many researchers in mathematics education (Good, Mulryan and McCaslin, 1992, Repo, 1996) and my main data, video recordings of the small-group discussions, fitted very well for that purpose. Later when developing the method of analysis I noticed that comparing the processes in the two different styles of the investigative approach appeared to be very difficult. I decided to focus solely on the class of the open approach and the two small-groups in that class. And adopting a general ethnographic approach seemed plausible to me.

During the experimental course and in the early phases of the analysis I saw two themes arising to me from the data. Firstly the investigative small-group approach was not just a matter of changing working methods. I felt some kind of "friction" in trying to make my students investigate mathematics and construct mathematical meaning by themselves. Remember the student who could not do observations from the data and draw conclusions! She was not necessarily stupid. Her behavior may be explained so that all her life in mathematics lessons she was made to imitate and apply the methods that were already taught to her. The requirement of constructing new mathematical knowledge did not fit her expectations of the roles of the teacher and the students. And it was not only the students who caused the problems. I myself had difficulties in acting consistently with the new approach. I went to read research literature and found Cobb and Yackel's (1996) concepts of social and sociomathematical norms. It seemed to me that they were writing about the same phenomena which I had experienced in my experimental classroom.

Another theme that arose to me from the data was the different styles of interaction of girls and boys in the original four small groups selected for video recordings. The students were allowed to choose their partners by themselves and almost all the groups in the two experimental classes consisted of girls or boys only. Before my final research questions were shaped I had written two papers (Partanen, 2005, Partanen, 2007) where I delineated the sociolinguistic subcultures of the girls and boys in those groups. These analyses could not, however, be conveniently included in the final research report.

1.4 About research on norms in mathematics classrooms

Up to now, norms in mathematics classrooms have been discussed in three different research traditions. Firstly, researchers in the interactionist tradition have studied implicit patterns of interaction in classrooms which are normative in the sense that they are taken for granted (Bauersfeld, Kurmmheuer and Voigt, 1988, Voigt, 1994). Building on and extending the interactionist tradition Cobb and Yackel (1996, 1998) developed a social constructivist approach, *the emergent perspective,* including an interpretive framework for analyzing important processes in inquiry mathematics classrooms. Main concepts of the framework are *social* and *sociomathematical norms* and *classroom mathematical practices* as well as their psychological correlates; *beliefs about one's own role, other's roles* and *the general nature of mathematical activity,* and *mathematical beliefs* and *conceptions*. The collective concepts, classroom norms and practices, are reflexively related to the psychological ones. Cobb and Yackel's concept of norm (Cobb and Yackel, 1996, 1998) is dynamic with the emphasis on explicit negotiation of norms compatible with collaborative student-centered problem solving approaches.

Secondly, research following the French epistemological/situated perspectives draws on Brousseau's concept of didactic contract (Brousseau1984, 1997). Teacher and students have tacit and mutual agreements that the teacher should know the content and is expected to teach it to the students and that the job of the students is to learn the content set out for her. Herbst (2003) emphasizes that the contract exists at different levels. One can either think about broader institutionalized relationships of teachers with students and subject matter or the details of the contract which are constantly negotiated at classrooms by teachers and students. The latter is close to classroom social and sociomathematical norms of Cobb and Yackel (1996, 1998). A third tradition where norms are discussed is socio-cultural perspectives. From outside the tradition it is difficult to discuss its ideas confidently, but one very important aspect is brought forth by the researchers in that tradition. They have worked to connect the theorizing about norms in classrooms to other cultural processes at different levels (Gorgorio and Planas, 2005b, Herbel-Eisenmann et al., 2003).

My project draws on the first tradition. I am building my analysis in the emergent perspective, but I shall also use ideas from the interactionist tradition. Cobb and Yackel (1996, 1998) emphasize that new norms should be negotiated for inquiry mathematics and they describe the transition from school mathematics tradition to inquiry mathematics. Their discussion of norms takes place in a situation similar to mine.

Different types of norms seemed to have relevance when I was constructing an account of the important processes in the experimental course. For many of the students the teaching experiment was the first time they studied mathematics by investigating in small-groups. The way of working clashed against many of our expectations. This way some traditional norms for the roles of students and teacher and for doing mathematics became visible. Through asking my students to discuss and create mathematical knowledge, for example, I started the implicit negotiation of new norms. And there were few situations when I initiated the explicit negotiation of new norms. Some of the norms discussed in my report are specific to the small-group phase of the lessons. Thus, my project intersects with research on the use of small-groups in instruction. But I shall also make interpretations about classroom norms by looking at the interactions in the two small groups. The aim of this research is to describe the ecology of social and sociomathematical norms in my upper secondary high level class beginning to study the basics of calculus by the investigative small-group approach. A further aim is to analyze how acting according to certain norms was intertwined with the occurrence of learning opportunities for the students. The aims are attempted to be reached through analyzing and interpreting the peer and student-teacher interactions in the two small-groups. Consistent with the ethnographic approach, other types of data, my acquaintance with the students and the school, my teacher's experience and my earlier analyses of the data were taken as a background against which I interpreted the observations from the video recordings.

Cobb et al. (Cobb, 1999, Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997, Gavemeijer, Cobb, Bowers and Whitenack, 2000, Cobb 2002) and researchers following them have set up long teaching experiments where they have applied the principles of realistic mathematics education and analyzed the negotiation of norms and emergence of mathematical meaning. Sometimes the research group has worked with a teacher for a whole school year. Together they have designed and revised hypothetical paths for collective learning called learning trajectories (Gravemeijer, 1994) and associated tools for certain mathematical topics. I see this kind of design research (Brown, 1992, MacClain, Cobb and Gravemeijer, 2000, Confrey and Lahance, 2000, Cobb, 2001) inspiring and very important in showing what is possible in classrooms. On the other hand, I want to emphasize that quite another job is for those ordinary teachers who strive for developing ways of applying problem-solving and investigative approaches in their instruction in naturalistic settings in different schools. We are not always such skillful teachers as the research reports describe and we have to start the developmental work from our particular situations. We are subject to numerous restrictions imposed on us by the school schedules and school cultures and we don't necessarily have any support in our attempts. To be able to negotiate norms and thus develop the cultures of our classrooms we may need to go into deep reflections about our own beliefs. We are part of the tradition and the tradition is part of us, more than we may understand. Teacher research discussing social and sociomathematical norms when teachers are developing their own practice in naturalistic settings is rare in the literature. Through my project I am trying to answer the challenge.

During the teaching experiment I did not know about the emergent perspective and research on norms. I did not see the need for negotiating new norms for the new way of studying. I was almost like any Finnish upper secondary mathematics teacher beginning to apply the investigative small-group approach in her teaching. During the last two or three years investigations as a means of teaching and learning content have appeared in Finnish upper secondary mathematics text books. Most probably an increasing number of teachers are beginning to use them, and an increasing number of students will confront investigations for the first time in their studies. My research may shed light on typical processes in such a situation and give valuable understandings and insights for teachers who wish to develop their practices in that direction.

2 THE EMERGENT PERSPECTIVE AND LOCALLY SITUATED VIEW OF LEARNING

2.1 Introduction

I situate my project in the landscape of social constructivism, more precisely in that of the e*mergent perspective* developed by Paul Cobb and Erna Yackel (1996). The primary defining characteristic of this framework is the coordination of interactionism and psychological constructivism (Cobb and Yackel, 1996, Cobb and Bauersfeld, 1995). During the last decades in mathematics education research, there has been an apparent opposition between on one hand traditions and theories emphasizing collective perspectives, like Vygotskian and activity theory traditions as well as the sociolinguistic tradition, and on the other hand the neo-Piagetian tradition which stresses the focus on the individual autonomous learner (Cobb and Bauersfeld, 1995). Social constructivist theorizing seeks to transcend the gap.

In this chapter I shall first give a short introduction to the background theories of the emergent perspective after which I am going to discuss the framework itself. Then, I shall make an account of norms research in mathematics education and finally I am going to introduce my important concepts and present my research questions.

2.2 Interactionism

Dissatisfied with the contemporary theories of mathematics education in the 70's and 80's which were either related to individual students, teaching or subject matter, Heinrich Bauersfled, Götz Krummheuer and Jörg Voigt (1988) developed an *interactional theory of learning and teaching mathematics*. Their main point is that social structures also contribute to the complexity of classroom processes. In their daily local interactions a teacher and her students form expectations and interpretations of each other, check these through the process of negotiation and thus jointly produce "meanings, structurings, and norms of acceptance and of validity". The reality of the classroom emerges from these interactions. In the ongoing processes of interaction and negotiation with other subjects, individual participants construct their personal interpretations which thus are "socially transmitted" (Bauersfled, Krummheuer and Voigt, 1988.)

The basis for the ideas in interactionism comes from sociological theories like symbolic interactionism, ethnomethodology and phenomenology, but some models from cognitive sciences were also used (Bauersfeld, Krummher and Voigt, 1988). According to the authors, common to all these theories is a certain constructivist perspective. When employing interactionism in their framework, Cobb and Yackel (1998) emphasize the roles of symbolic interactionism (Blumer, 1969) and ethnomethodology (Mehan and Wood, 1975) in it.

2.3 Radical constructivism

Based on the writings of Piaget, Ernst von Glasersfeld (1989, 1995a) developed his theory of knowing and learning *radical constructivism*. Piaget's notions of assimilation, accommodation and viability of knowledge, as well as the rejection of the correspondence theories of truth are included in this perspective which can be characterized as both epistemological and psychological. According to Glasersfeld, perturbations that the cognizing subject generates, relative to a purpose or goal, give the driving force to development. Learning is characterized by a process of self-organization, in which the subject reorganizes her activity in order to avoid perturbations and to reach conceptual equilibrium. According to Glasersfeld (1995a) the most frequent source of perturbations is the interaction of the subject with others. The focus of radical constructivism is on the construction of conceptual knowledge Glasersfled (1995b).

2.4 The emergent perspective

2.4.1 Coordinating complementing perspectives

The anticipation of the need for integrating or complementing radical constructivism with interactionism is already seen in the writings of Bauersfeld (1988, 1995). But it was his colleagues Cobb and Yackel (1996), who in a more explicit way presented the new framework.

Cobb and Yackel (1996, 1998), describe how the emergent perspective was developed along a series of teaching experiments where they were seeking to develop mathematics education in primary schools towards inquiry mathematics. Initially the researchers had a constructivist position and their aim was to account for the learning of individual students when they, based on their experiences, reorganize their thinking and thus develop more sophisticated ways of knowing mathematics. In a co-operation project with European researchers in mathematics education Cobb, Yackel and Wood (Cobb, 1995) realized that in classrooms, where students interact with a teacher and their peers, the students' constructions have an intrinsically social aspect. The social constructivist or emergent perspective, evolved from the reflexive interplay of theory and practice in their developmental research. Cobb and Yackel emphasize that they want to avoid the essentialist assumption that their framework might capture the structure of individual and collective activity independent of history, situation and purpose (Cobb and Yackel, 1996).

The interactionist perspective developed by Bauersfeld and his colleagues can be seen to complement the psychological constructivism. According to Bauersfeld (1988), constructivists think that

> the subjective construction of reality leads to a viable adaptation to the resistance and to the obstacles of a world, which the subject can describe and understand only via these constructions. (p. 39)

But, Bauersfeld continues, on the other hand

> the descriptive means and the models used in these subjective constructions are not arbitrary or retrievable from unlimited sources, as demonstrated through the unifying bonds of culture and language, through the intersubjectivity of socially shared knowledge among the members of social groups, and through the regulations of their related interactions. Apart from the adaptation to the resistance of "the world", learning is characterized by the subjective reconstruction of societal means and models through the negotiation of meaning in social interaction and in the course of related personal activities. (p. 39)

Thus Bauersfeld (1988) sees two ways of constructing knowledge. According to him the "subjective structures of knowledge are subjective constructions functioning as viable models, which have been formed through adaptations to the resistance of "the world" and through negotiations in social interactions" (p. 39). The separation of the two ways may be necessary for analytical purposes, but, Bauersfeld stresses, one should not lose sight of their fundamental inseparability (Bauersfeld, 1988).

Cobb sees the complementing of the perspectives more dramatically. Already in 1989 he wrote about the relationship between experiental, cognitive and anthropological (sociological) perspectives in mathematics education research (Cobb, 1989). For him "complementarity is an expression of the apparent paradox between seemingly opposite positions" (p. 41). In our lives, in teaching and in science too, we have to cope with complementaries as between emphasizing individual students' understanding and covering the curriculum, or for example between thinking of light as waves and particles. Cobb stresses that "we cannot resolve the problem of complementarity once and for all. Rather we have to learn to cope with it in local situations" (p. 41). Cobb (1989) calls for analyses that coordinate experiential, cognitive and anthropological contexts while at the same time considering them non-intersecting domains of interpretation, complementary though irreducible. Let the previous ideas illuminate the idea of complementarity, which most probably has been behind developing the emergent perspective. In a more recent paper Cobb et al. (2001) describe the relationship between the social and psychological perspectives as follows:

> each perspective constitutes the background against which mathematical activity is interpreted from the other perspective. ... When we take a sociological perspective, we therefore locate a student's reasoning within an evolving classroom microculture, and when we take a psychological perspective, we treat that microculture as an emergent phenomenon that is continually regenerated by the teacher and students in the course of their ongoing interactions ... the coordination is between two alternative ways of looking at and making sense of what is going on in the classrooms. (p. 122)

Cobb and Yackel have been criticized for using both radical constructivism and interactionism in their framework. Speaking from the sociocultural tradition Lerman (1996) argues that the radical constructivist position has coherence in that it denies the value of talk about the real world. It does not have, and it doesn't need, an explanatory model of how knowledge becomes internal. This is because any sense of knowing, other than internal, is excluded in the perspective. But, Lerman stresses, it is incompatible to talk about internalization or enculturation in the context of radical constructivism.

> As long as the gaze is on the individal mind, with meaning-making being interpreted as the sense an individual makes of her or his experiences, one has to continue to reject the possibility of match between the conceptualizations of one person with another or between the individual's representations and reality. (p. 141)

In line with Lerman, Waschescio (1998) criticizes the approach of Cobb and Yackel which tries to integrate social processes into the epistemological framework of radical constructivism. He argues that their concept of social interaction doesn't provide an explanation for the occurence of learning in the classroom. This is because of their way of distinguishing between two levels of meaning, namely an individual and social one and the rejection of the concept of internalization. The problem is that, on the interactional level, meaning is conceptualized as the result of negotiation. But negotiation results in nothing more than meanings taken-to-be-shared. It is not necessary, in this framework, that students and teacher share knowledge. The interactionally established meanings do not become part of the individual students' cognitive repertoire (Waschescio, 1998.) According to Waschescio (1998), the meanings taken-to-be-shared do not achieve the status of cognitive element nor constitute a social element in the sense of a commonly shared structure. He claims that, after all, according to Cobb, the source of learning is always in the individual.

Cobb, Yackel and Bauersfeld do not agree with their criticizers. In their view interactionism is compatible with psychological constructivism (Yackel, 2001, Cobb and Bauersfeld, 1995), but in their writings they don't too often go into the details of the justification but leave the judgement to the reader. Cobb (2000) answers Lerman and Waschescio by explaining the nature of the relationship between the two perspectives to be reflexive. He writes that the two perspectives are not only interdependent, but neither perspective exists without the other. Other researchers have also commented on the criticism of Lerman (Steffe and Thompson, 2000). To me it seems that Lerman and Waschescio haven't considered seriously enough the concept of *interpretation* in symbolic interactionism (Blumer, 1969, p. 5). According to Blumer the source of meaning is on the social level, but the meanings are not internalized as such. Using the meaning by a person in his or her action involves interpretation, which is a formative process where meanings are selected, checked, suspended, regrouped and transformed in the light of the situation and direction of the action (Blumer, 1969). In my view, the concept of interpretation is compatible with a radical constructivist view of a subject constructing knowledge from her experience. And I think that it does establish a link between meanings on the social level, which an observer can recognize in the joint action (Blumer, 1996) of a group of people, and personal meanings of an individual subject.

It also seems to me that Lerman and Waschescio are taking a more essentialist relation to theory than Cobb and Yackel, who rather emphasize the usefulness of their framework when trying to understand what is going on in classrooms. Cobb (1994) argues for pragmatic use of theories. A particular perspective can be chosen for the reason that it works better than another for a given purpose. According to him pragmatic justifications reflect the researcher's awareness that he has adopted a particular position for particular reasons. Also, a pragmatic approach to theorizing brings with it the possibility of coordinating perspectives while addressing specific problems and issues (Cobb, 1994). According to Cobb and Yackel (1996) the main idea in the emergent perspective is the coordination of analyses of classroom processes which are made in psychological and sociological terms.

2.4.2 Interpretive framework

Voigt (1996) argues that the symbolic interactionist perspective is useful when studying students' learning in inquiry mathematics classrooms because it emphasizes both the individual's sense making processes and the social processes without giving primacy to either one. In the emergent perspective, individual students are seen as actively contributing to the development of the classroom microculture that both allows and constrains their individual activities. Neither individual student's mathematical activity nor the classroom microculture can be adequately accounted for, without considering the other. Whereas, in the Vygotskian theories, the cultural and social are given primacy, in Cobb and Yakcel's perspective an individual student's activity and the classroom microculture are reflexively related (Cobb and Yackel, 1998.)

Cobb and Yackel have developed an interpretive framework for analyzing individual and collective activity at the classroom level (1996).

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, other's role, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1 An interpretive framework for analyzing individual and collective activity at the classroom level (Cobb and Yackel, 1996, p. 211)

The interactionist perspective is used to analyse sociological constructs like *classroom social norms* and *sociomathematical norms*. The psychological correlates of social norms are the interacting persons' *beliefs about their own role, other's role* and *the general nature of mathematical activity at school*. Correspondingly the individual counterparts of sociomathematical norms are the students' and the teacher's *mathematical beliefs* and *values*. In each of the two first rows of the table, norms and beliefs are seen as reflexively related so that neither exists independently of the other (Cobb and Yackel, 1996, 1998.)

The third aspect of the social perspective in Cobb and Yackel's interpretive framework is *classroom mathematical practices*. As well as one can think of an individual child's mathematical development, one can also consider the learning of a classroom community. Ways of interpreting number words and numerals, for example, may at some point become taken as self evident by a primary classroom community, and constitute a classroom mathematical practice. Evolving mathematical practices, viewed against the background of classroom social and sociomathematical norms, can be seen to constitute the immediate, local situations of a student's development (Cobb and Yackel 1996). There is, however, no direct link between classroom mathematical practices and the individual student's mathematical conceptions and activity. Instead, children's participation in classroom mathematical practices constitutes conditions for the possibility of mathematical learning. Classroom mathematical practices and individual students' *mathematical conceptions and activity* are reflexively related (Cobb and Yackel, 1996, 1998.) Cobb and Yackel (1996) clarify the nature of the different analyses:

> In general, analyses conducted from the psychological constructivist perspective bring out the heterogeneity in the activities of members of classroom community. In contrast, social analyses of classroom conducted from the interactionist perspective bring out what is jointly established as the teacher and students coordinate their individual activities. (p. 212)

It is important to notice that the social in the interactionist perspective is not the same as the socio-cultural in most of the Vygotskian theorizing. According to Skott (2004) the social was used by Vygotsky in his writings in two different senses. First, it was used in the socio-cultural sense that describes meaning as socially constituted. Scientific concepts, for example, carry with them social meanings that allow the individual to adopt certain perspectives on the world. Secondly, Skott argues, that the social was also used by Vygotsky in a local sense, in the sense of human interaction. Cobb and Yackel (1998) emphasize that "Bauersfeld, however, takes the local classroom microculture rather than the mathematical practices institutionalized by wider society as his primary point of reference" (p. 161).

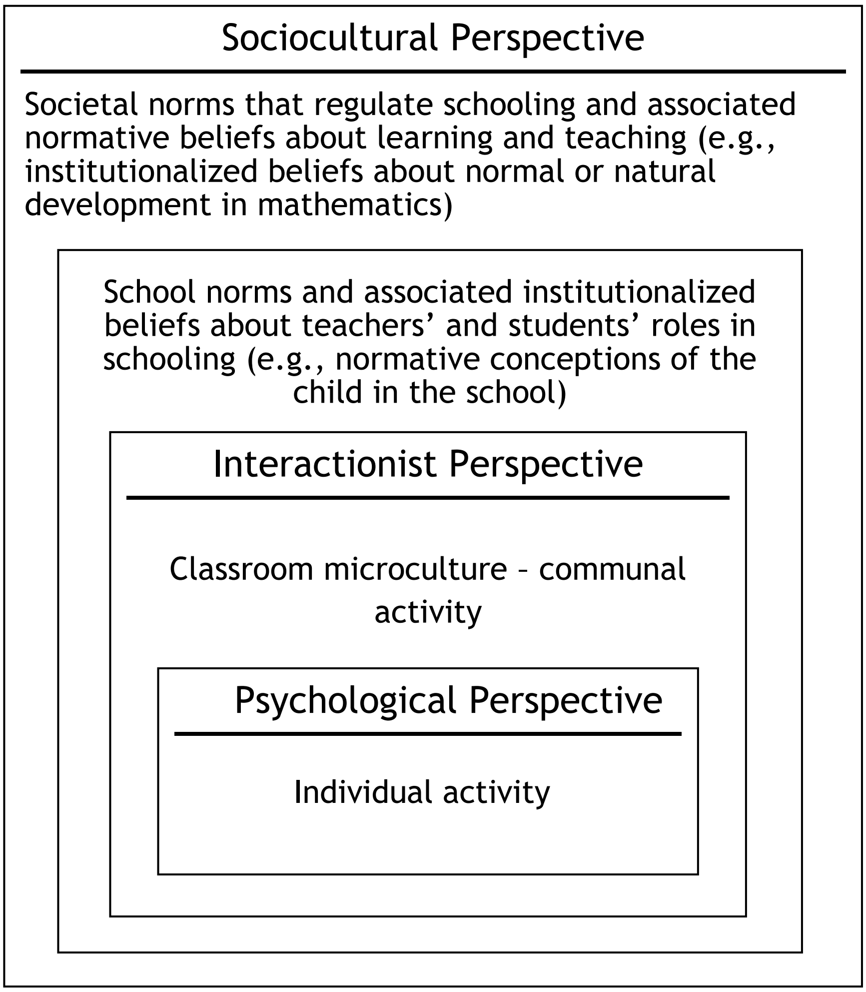


Figure 2 An elaboration of the interpretive framework (Cobb and Yackel, 1996, p. 216)

Cobb and Yackel (1996, Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997) explain that in their research and developmental work they very often could develop explanations adequate for their purposes by referring to the classroom processes only. But there were, however, occasions when they found it essential to take account of the broader institutional contexts in which such systems are embedded. They present an elaborated version of their interpretive framework (figure 2). They see that the psychological framework is embedded in the interactionist perspective describing that from the interactionist perspective students' individual activities are framed in terms of their participation in the practices of their classroom community. Conversely, participation in these practices constitutes the collective background. The psychological perspective and the interactionist perspective together make what Cobb and Yackel call the emergent perspective. This is further embedded in the socio-cultural framework. When taking account of norms and practices at the school and societal level, Cobb and Yackel found themselves adopting a socio-cultural perspective. They emphasize that the interactionist and psychological perspectives are not subsumed by the socio-cultural perspective. The key for coordinating socio-cultural and emergent perspectives, is recasting the global process of appropriation from the socio-cultural environment as one of negotiation and individual construction at the classroom level (Cobb and Yackel, 1996). The precision of the emergent perspective is appropriate for certain purposes. In other situations the global nature of socio-cultural accounts has its own advantages (Cobb and Yackel, 1996.)

Through the examples Cobb et al. give (Cobb and Yackel, 1996, Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997) about their use of a socio-cultural perspective, it seems to me that they write about institutionalised contexts of schooling at different levels. But they seem to exclude ideas like that students, as participants of different cultures and communities, may bring different norms for social interaction into the classroom, or that different norms might be negotiated for different students in the classroom (Planas and Gorgorio, 2004).

2.5 Social and sociomathematical norms

Social and sociomathematical norms are important concepts which the authors of the emergent perspective brought into wide discussion in mathematics education research. They also have their correlates in other theoretical traditions. In this section I am going to present how the concepts are understood in the emergent perspective. Then I shall make a general account of research on norms in mathematics education. And finally I am going to discuss the use of the concepts in this research.

2.5.1 Social and sociomathematical norms in the emergent perspective

For Cobb and Yackel (1996) classroom social norm is a sociological construct. In their first teaching experiment when they tried to build up an inquiry mathematics classroom it became apparent that the students were not accustomed to explaining how they had interpreted and attempted to solve certain tasks. In their previous school year the students had participated in a traditional classroom where they had been “steered toward officially sanctioned solution methods”. It had been the students’ task to try to find out what the teacher had in mind rather than express their own understandings. The teacher in the teaching experiment started the renegotiation of classroom social norms. In the course of the events in the classroom, in the interactionally constituted situations, she discussed with her students the obligations she expected them to fulfil. The new norms grew from the joint activity as the teacher and students did and talked about mathematics (Cobb and Yackel, 1998, Yackel and Cobb, 1996.) Methodologically speaking, social norms “characterize regularities in communal or collective classroom activity and are considered to be jointly established by the teacher and students as members of the classroom community”(Cobb and Yackel, 1996, p. 212).

The topics of explicit negotiation of social norms between the teacher and the students in Cobb and Yackel’s first experiment were, for example, that the students were expected to explain and justify their solutions, to attempt to make sense of others’ explanations, to indicate agreement and disagreement and to question alternatives in the situations of conflict (Cobb and Yackel, 1998). Generally speaking, in the emergent perspective, social norms are seen to sketch the classroom *participation structure* (Cobb and Yackel, 1996). Cobb and Yackel (1996) refer to Lampert (1990) and Erickson (1986), and Lampert, for her part, writes that the concept of participation structure is taken from the unpublished dissertation of Florio (1978) and refers also to Ericson and Shultz (1981).

It seems to me that there is ambiguity in research literature on the concept of participation structure. Lampert (1990) writes about *classroom participation structure*, which is a construct describing the culture of the classroom, a collection of patterns of interaction. This is also the case of Erickson (1986): “In interaction in school lessons a dimension of culturally patterned social organization (patterns for turn-taking, listening behaviour, and the like) always coexist with the logical organization of the information content of the subject matter” (p. 136). Erickson and Schultz (1981) join this thread of argument when writing about how the capacity of monitoring contexts is an essential feature of social competence. They connect *participant structure* to a context. On the other hand, Erickson (1992) gives an introduction to mircoethnographic analysis of interaction. When explaining the issues of data analysis and writing about identifying major constituent parts of events, he describes two aspects of relationships of participants.

Physical relations, like posture, shared gaze and interpersonal distance, for example, define the patterns of physical relations, but relational roles, social identities and rankings are aspects of the overall pattern of social organization, the *social participation structure*. Usually, from one major constituent to the next, there is a rearrangement of both physical relations and social participation structure (Erickson, 1992.) Here, it seems to me, the concept is more concrete, something that can be analysed from a fragment of interactional data.

There is ambiguity also in the content of the concept of participation structure. The examples that Cobb and Yackel (1996, 1998) give about social norms seem to me to be describing ways of communicating, ways of participating in classroom discussions. Erickson and Schultz' (1981) "ways of speaking, listening, getting the floor and holding it, and leading and following" (p. 148) come close to it. But the latter researchers also have an element of the style of co-operating when they mention getting and holding the floor, leading and following. Erickson (1992) includes in participation structure "relational roles, social identities and rankings". This reminds me of what Cobb and Yackel (1996, 1998) give as psychological correlates of social norms: beliefs of one's and other's roles. Lampert (1990) emphasizes that participation structure defines the roles and responsibilities of both teacher and students in relation to learning and knowing. This idea is also discussed by Cobb and Yackel (1998, p. 167) and it could perhaps be seen to be included in "beliefs about the general nature of mathematical activity at school".

In the emergent perspective, beliefs, the psychological correlates of the classroom social norms are in a reflexive relationship with the norms: "individual interpretations that fit together constitute social norms that both allow and constrain the individual interpretations that generate them" (Cobb and Yackel, 1998, p. 168).

The focus on norms was not, for Cobb and Yackel, an end in itself. They wanted to account for students' mathematical development as it occurs in the social context of the classroom. Social norms can, however, be applied to any subject (Cobb and Yackel, 1996.) The authors saw this as a limitation and they transcended their analyses to include the normative aspects of whole-class discussions that are specific to students' mathematical activity (Lampert, 1990, Voigt, 1995, Yackel and Cobb, 1996). In their experimental classrooms, Cobb and Ycakel wanted to enhance student autonomy by developing "communities of validators" such that were established by means of mathematical argumentation rather than by appealing to authorities like the teacher or the textbook. But, they explain, for this to happen it was not sufficient that students know that they are expected to make mathematical contributions. To avoid mathematically unproductive discussions, students should develop a sense of knowing in action when it is appropriate to make a mathematical contribution and what constitutes an acceptable contribution. Students should be able to evaluate themselves, for example, what counts as a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution and an acceptable mathematical explanation (Cobb and Yackel, 1996, Yackel and Cobb, 1996.)

Cobb and Yackel saw that in their inquiry mathematics classrooms students constructed specifically mathematical beliefs and values that enabled them to act as increasingly autonomous members of classroom mathematical communities. These beliefs are seen as psychological correlates of classroom sociomathematical norms and they are reflexively related with the norms (Cobb and Yackel, 1996.)

2.5.2 Norms in mathematics education research

The increasing interest of researchers in mathematics education abot social processes and learning has led to investigations on how patterns of interaction become normative. Researchers from multiple perspectives have contributed to studying this topic. Norms have been approached among others from
a) sociological (interactionist) and emergent perspectives
b) epistemological/situated perspectives and c) sociocultural perspectives. The importance of the concept "norm" and the way norms are analysed varies according to tradition, but common to all of them is to divide normative behaviours to more general processes in the classroom and to those tied closely to the specific content under study (Herbel-Eisenmann et al., 2003.) In the following I am going to discuss the definition of norms in different perspectives and give some examples of research reports made in each tradition. I do not claim my discussion is all encompassing. The field is vast overall in the world as well as inside each tradition.

Sociological and emergent perspectives

Researchers in the interactionist tradition seldom use the term norm, but nevertheless, an idea of normative interactions is seen in their theorizing. In their everyday interactions teacher and students develop certain routines for the smooth functioning of classroom discourse. Otherwise the complexities of a classroom would be too hard to accommodate. Routines are connected by interactional obligations; a routine action is followed by expectations of another routine action as a reaction and so on. The network of routines and obligations can be described as patterns of interaction. These patterns are produced interactively and they become normative in the sense that they are taken for granted (Bauersfeld, Krummheuer and Voigt, 1988, Voigt, 1994.) The developed patterns of classroom interaction form an "underlying grammar" of classroom processes. This is to describe that the regulations and norms are produced unconsciously and the participants are not aware of them. The persons just act as if they followed certain structuring. Therefore ethnomethodological description fits better with dealing with the phenomenon than sociological theories of action which speak about rule-guided behaviour and role taking of persons (Bauersfled, 1995.) According to researchers in the interactionist perspective, traditional routines and structurings in classrooms are very stable and difficult to change (Voigt, 1994).

Some routines that have been recognized in classroom interactions are: teacher's use of open questions to which one definite answer is expected, teacher's suggestive hints, the decomposing of a solving process into smaller units of subsequent actions, students' routines of answering in a trial-and-error method, watching the teacher's suggestive hints and verbal reduction which means restriction of utterances to numbers or catchwords (Bauersfeld, Krummheuer and Voigt, 1988). Patterns of interaction observed are, for example, the elicitation pattern and the discussion pattern of interaction. In the elicitation pattern the teacher proposes an ambiguous task, and the students offer differing answers which the teacher evaluates. If the students' contributions are too divergent, the teacher guides the students towards one definite solution. The teacher poses small questions and elicits bits of knowledge. After getting to the solution the teacher and the students reflect and evaluate the process. In many studies, the elicitation pattern is recognized as a typical pattern in mathematics classrooms (Voigt, 1995.)

While patterns of interaction are not specific to mathematics classrooms, and therefore they could as well be recognized in other classrooms, the thematic patterns of interaction are more specific to mathematics (Voigt, 1989). For example, in the thematic pattern of direct mathematization, a story or picture is directly interpreted as a certain calculation problem and alternative interpretations are not discussed or they are ruled out (Voigt, 1995).

The researchers in the interactionist tradition seem to be more interested in describing the existing norms rather than in investigating how the norms are established. Their discussions of norms are more "snapshot" compared to the way researchers in the emergent perspective show establishment and evolving of norms in the interactions of the classroom. In the interactionist perspective negotiations of norms are implicit, but the researchers in the emergent perspective emphasize the possibility and importance of making the negotiation of social and sociomathematical norms explicit to the students. Both the interactionist and emergent perspective focus on the microculture of the classroom, the broader cultural contexts, or the macroculture, are not really included in the analyses (Herbel-Eisenmann et al., 2003.)

I have found one recent research report done on the interactionist perspective. Johansson (2007) recognized the funnel pattern to be a standard pattern in the teacher-student interactions of a Swedish secondary school teacher when the class was working with exercises on an individual basis. In the funnel pattern a teacher, who recognizes a student with difficulties, guides her through step-by-step reduction of demands towards a right reaction which then could be interpreted as a sign for the teacher to present the solution (Bauersfeld, 1988).

There were progressive studies very much like research in the emergent perspective even earlier than Cobb and Yackel (1996) published their most important article. Magdalene Lampert (1990) tried, through action research in her

fifth grade classroom, to find whether it would be possible to change the practise of knowing mathematics in school to be more like doing mathematics in the discipline by altering the roles and responsibilities of teacher and students in classroom discourse. She found evidence that her students learned to do mathematics in a way which is more congruent with the disciplinary discourse. In a later article, Lampert et al. (1996) discuss the problems of constituting a practise of mathematical argumentation. In particular, disagreeing with others seemed to be difficult for her students. Lampert et al. argue that "bringing about the kind of social climate in schools that supports academic argument requires a major shift in cultural norms"
(p. 759).

The emergent perspective of Cobb and Yackel (1996) has inspired a wide range of research on norms. In their numerous papers the authors themselves have described the development of social norms (Cobb, Yackel and Wood, 1989, Cobb, Wood and Yackel, 1993) and sociomathematical norms (Yackel and Cobb, 1996, McClain and Cobb, 2001) in many teaching experiments in elementary classrooms and how these new norms can facilitate rich ways of learning. Yackel and Rasmussen (Yackel and Rasmussen, 2002, Yackel, 2001) undertook a teaching experiment in a university level differential equations class. By using the constructs of social and sociomathemathematical norms they analysed, among other things, the development of explanation, justification and argumentation in the class.

Many other researchers in mathematics education have taken up the important concepts of the emergent perspective. A study undertaken by Pang (2001) suggests that negotiation of social norms may be necessary, but not sufficient for fruitful mathematical learning. She compared two classrooms that had similar student centered participation structures but different sociomathematical norms. She emphasizes the importance of students' participating in specifically mathematical ways of explanation, justification and argumentation. Kazemi and Stipek (2001) have reported similar results after analysing norms in four elementary classrooms. They concluded that in spite of similar social norms in the classrooms there was a "high press" for conceptual thinking in those classrooms where there also was a focus on sociomathematical norms.

Hoster (2006) pointed to the importance of helping teachers in their attempts to support intellectual autonomy without relinquishing control of discourse in inquiry classrooms. She sees that the concepts of social and sociomathematical norms could enhance the pedagocial tool used in their research. Edwards (2007) argues that friendship groups of 11 – 15 year olds in her study offered an opportunity for certain sociomathematical norms to be negotiated effectively. Tatsis and Koleza (2008) investigated social and sociomathematical norms in the interactions of pairs of pre service elementary teachers doing problem solving in Greece. In their project they also studied the relations of norms with construction and negotiation of mathematical concepts.

Drawing from the ideas developed by Cobb and Yackel, Herbel-Eisenmann (2000, 2002), Hamm and Perry (2002) and Ju and Kwon (2007) have investigated discourse patterns connected to patterns of authority in mathematics classrooms. For example Ju and Kwon show, how during an inquiry-oriented differential equations class in a Korean university, the students' discourse changed from third person perspective to first person perspective. This was taken as a sign of students' positioning themselves as active mathematical inquirers. According to Herbel-Eisenmann et al. (2003) this type of research deals with a more specific set of norms rather than the broader category of social norms.

There are signs about the different importance or temporal appearance of different norms when teachers and researchers work to establish inquiry cultures in classrooms. For example the social norm that students are expected to explain their solution methods and mathematical thinking seems to be a logical basis for the development of the sociomathematical norm concerning what counts as an acceptable mathematical explanation and justification (Yackel and Cobb, 1996). Describing the evolution of explanations and justifications in their experiments, Yackel and Cobb (1996) write that, students first had to learn that explanations should have a mathematical, rather than social, basis. Later their students distinguished between different types of explanations, for example between those that describe procedures and those that describe actions on experientially real mathematical objects. Finally, some students started taking explanations as an object of reflection. McClain and Cobb (2001) describe how the norm of mathematical difference was a basis for the development of the norms of sophisticated and efficient solutions of tasks. Tatsis and Koleza (2008) describe the role of norms in the thematic development in their small groups. They point to the vital role of the collaboration norm since it was involved in all the stages of the thematic development.

One problem in the field is the variety of ways that researchers analyze and write about norms. Rasmussen and Stephan (2008) have developed a special methodology for documenting normative ways of reasoning of a classroom community. Their methodology might be applicable to analysing social and sociomathematical norms, which in the emergent perspective are seen as evolving sociological constructs.

Epistemological/situated perspectives

Brousseau (1997) discusses didactique, the science about the creation and articulation of knowledge, and about situations. Therefore his theory is labelled as being epistemological and situational (Herbel-Eisenmann et al., 2003.) The normative nature of interactions in the classroom between the teacher and students is included in Brousseau's concept of didactic contract (Brousseau 1984, 1997). The contract determines what are the responsibilities of one partner to another in teaching and studying mathematics. In short, Brousseau defined it as the tacit, mutual understandings that the teacher knows the content and is expected to teach it to the students. Conversely, the student understands that the teacher knows the content and it is her job to learn the content that is set out for her. Brousseau (1984) points to the paradoxical demands of the contract. Everything that the teacher does “to make the pupil produce the behaviours he expects, tends to deprive the latter of the conditions necessary for understanding and learning the notion concerned” (p. 113). And, from the perspective of the student: “If he accepts that the teacher, according to the contract, teaches him the results, he will not attain them himself and thus will not learn mathematics, i.e. will not make mathematics his own” (p. 113). Brousseau contends that “learning will not be based on the correct functioning of the contract, but rather on breaching it” (Brousseau, 1984, p. 113.)

It seems to me that Brousseau, at least in his early writings (1984), is speaking about one specific contract, the Contract, but, that later researchers, as for example Blomhöj (1994) and Herbst et al. (2007), have applied the concept in the sense that it is seen as a set of negotiable norms in the local context of teaching and studying mathematics at school. Herbst (2003) clarifies the difference by writing about a more global, large print of the contract, about broader institutionalized relationships of teachers with students and subject matter, and continues that Brousseau postulates that the specific clauses of such a contract – the fine print – are permanently being negotiated at the microlevel by teacher and students as they do their share of work (Herbst, 2003).

According to Herbel-Eisenmann et al. (2003), the didactic contract of Brousseau (1997) carries with it aspects associated with social norms as well as with sociomathematical norms, but it also appears to be centrally tied to particular mathematical ideas being studied and thus has characteristics similar to classroom mathematical practices. Herbst (Herbst et al., 2007) has developed a systemic approach to instructional situations. He distinguishes three basic elements that are in interaction with each other: teaching, studying and mathematics. He hypothesizes that norms regulate the relationships between each component, and defines that the set of those systemic relationships is the didactic contract. In the tradition following Brousseau, the normative aspects of classrooms are considered to be implicit, they are tacitly understood but can not be ignored (Brousseau, 1997). One difference between social and sociomathematical norms and the didactic contract is that the former are viewed to be mutually negotiated within the microculture of the classroom, whereas the latter is seen to be both pre-existing and as existing in particular situations (Herbel-Eisenmann et al., 2003).

In the tradition following the work of Brousseau, Herbst (2003, Herbst et al. 2007) has investigated teacher’s practical rationality in teaching experiments in order to increase understanding of school reforms from the perspective of the work of the mathematics teacher. Novotna and Hospesova (2007) studied in a Check classroom how the use of the Topaz effect (Brousseau, 1984, 1997) reflected teachers’ beliefs and how it influenced students’ work. They concluded that the frequent use of this pattern decreases students’ responsibility for successful completion of mathematical problems. The pattern made lessons run smoothly, but students were losing self-confidence, were becoming teacher-dependent and they felt that they did not understand mathematics. Biza, Nardi and Zachariades (2009) investigated connections between teachers' beliefs about visual arguments in mathematics, their sufficiency and persuasiveness, and the nature of the didactic contract they offered to their students.

Sociocultural perspectives

Saxe (1999a, 1999b, 2001) describes collective practices and activity structures that play a role in mathematical goals. Activity structures can be used to interpret the goals that emerge for individuals in collective practices and include:
a) routine phases or cycles of activities, b) norms and sometimes explicit rules for behaviour and c) emerging role relations between participants. Herbel-Eisenmann et al. (2003) argue that although Saxe did not specifically define the term norm in his work, he (Saxe, 2001) is describing norms that occur within the activity systems of a mathematics classroom. However, it is not as central for him as the notion of collective practice. For example, a teacher who values recent NCTM recommendations would have a different kind of practice than a teacher that values a procedural approach (Herbel-Eisenmann et al., 2003.) In the socio-cultural tradition, as in the work of Saxe (e.g. 2001), norms are described in an instant of time rather than in a developmental way (Herbel-Eisenmann et al., 2003). Planas and Gorgorio (2004) claim that even during the change of the millennium, the study of norms from a socio-cultural perspective was only in its beginning stages.

Herbel-Eisenmann et al. (2003) notice that some socio-cultural approaches seem to take the mathematical understandings, the microculture of the classroom, the broader macroculture etc., all into account. Here lies the criticism that Gorgorio and Planas (2005a, 2005b) express towards the emergent perspective. In their research on multiethnic mathematics classrooms in Spain, they found the concept of norm very essential. But they noticed that immigrant students had a different way, different from that of the local students and the teacher, of understanding, valuing and using mathematics. The meanings and values associated with mathematical knowledge and who is mathematically knowledgeable, the expected role of a mathematics teacher, and so on were by no means taken-as-shared. According to Gorgorio and Planas (2005a, 2005b), from a socio-cultural perspective, all participants of the classroom community must be seen as social individuals with their own social and cultural experiences and expectations. Also Herbel-Eisenmann et al. (2003) point to the need of accounting for other communities of practice in which the participants are involved instead of just focusing on the micro-processes of a classroom.

Gorgorio and Planas (2005b) reconstruct the concept of norm as cultural artefact mediating classroom interaction:

> We understand norms as being secondary cultural artefacts as defined by Cole (1996). Sociomathematical norms are shaped by cultural schemas, representations and valorisations of mathematical knowledge and its ownership. They regulate and legitimise interaction and communication processes of mathematical practice. Norms of mathematical practice as interpretations of cultural schemas about what mathematics in schools is/should be about, regulate the content of practice as legitimised within the classroom. (p. 70)

Their concept of sociomathematical norms overlaps with some social norms of Cobb and Yackel and the concept of norms of mathematical practices is close to Cobb and Yackel's sociomathematical norms (Gorgorio and Planas, 2005b).

Some recent research reports from the socio-cultural tradition seem to blur the borderlines of the emergent perspective and socio-cultural approaches. Hunter (2007) reports the interactional strategies a teacher used to constitute a classroom context in which students participated in the discourse of collective argumentation. She found that norms of sense-making were established in the community and that the metaphor of convincing provided motivation for the students to engage in the development of argumentation. According to Hunter, even young children can participate in collective argumentation when carefully scaffolded.

Tsai (2007) designed a study where two researchers co-operated with a teachers' professional community in order to develop norms for their classroom communities, such that the students were willing to engage in discourse. A relationship was found between enhancing teachers' autonomy in their teaching practice and the students' intellectual and social autonomy.

Piatek-Jimenez (2005) studied an advanced undergraduate classroom. She emphasizes the importance of considering the role played by the norms and practices of the mathematics (professional) community in the development of the classroom norms and practices. She defines the terms "community social norms", "community sociomathematical norms" and "community mathematical practices" to refer to the norms of the professional community of mathematicians.

2.5.3 Social and sociomathematical norms in this research

In my project, I see, in line with Cobb and Yackel (1996), that classroom social and sociomathematical norms are sociological constructs. If in a certain context the expectations (beliefs about how things should be) of individuals for their own and others actions fit together, they make a norm. Norms and individual persons' expectations are reflexively related. In my project the aim concerning norms is to describe the ecology of social and sociomathematical norms in my upper secondary class when I introduced the investigative small-group approach. For this purpose, I needed ideas from both the interactionist and the emergent perspectives. The interactionist theorizing emphasizes an ethnomethodological approach (Mehan and Wood, 1975, Peräkylä, 1990) for finding patterns of interaction through which norms are produced (and reproduced) and implicit mutual negotiation (Bauersfled, 1995). Cobb and Yackel (1996, 1998) give a stronger role to the explicit negotiation and development of new norms. I see that the expectations of individuals and production or negotiation of norms may either be implicit or explicit; individuals may be or may not be aware of them. However, with both traditions, it is compatible to define classroom norms as regularities in communal or collective classroom activity, which are considered to be jointly established by the teacher and students (Cobb and Yackel, 1996).

I am thinking of the concept of participation structure at two levels. One level is the culture of the classroom. Social norms together make up the classroom participation structure (Lampert, 1990, Cobb and Yackel, 1996). On the other hand, in my analysis I found it useful to apply the concept in a more concrete sense as the *episode (social) participation structure*, which can be analysed in actual interactions of the small-groups during the episodes in my data (Ericson, 1992). In the research report I am mostly referring to the latter meaning. I wanted to be open to the different aspects of participation structure; however, so that they were relevant in terms of my goals and my data. Three features of the concept appeared to be of special importance for my research. On the one hand I am interested in "ways of speaking, listening, getting and holding the floor, leading and following" (Erickson and Schultz, 1981, p. 148), co-operating and the roles that the students and I took in these activities. These might be characterized as 1) participation in communication and 2) participation in working together. On the other hand, because of the nature of the investigative approach, it seemed important to include in the analysis 3) the roles of teacher and students in relation to knowing and learning (Lampert, 1990).

In my analysis I searched for regularities and interesting occurrences in the participation structures jointly established by the students and me. Based on them, I made interpretations of social norms being produced or negotiated. I also looked for the production and negotiation of sociomathematical norms by searching for expressions or acting of mathematical beliefs and values in our discussions.

All the participants in my teaching experiment had a long history of experiencing Finnish school mathematics and of studying mathematics in our school. The norms established in traditional classrooms form the initial situation in which we started the teaching experiment. Because of the short length of the experiment and the fact that no conscious negotiation of new norms occurred, one has to expect that some old norms were there all the time. Occasionally traditional norms were compatible with my expectations in the new situation, but very often they contradicted the investigative small-group approach. My negotiation of new norms was mostly implicit. By giving investigative tasks to my students and by asking them to work in small-groups with the assistance of technology, I started negotiations of new norms and very often the students acted according to my expectations. It happened also, though very rarely, that I initiated the negotiation of new norms in an explicit way and the students joined in acting according to my expectations. Germs of new norms could be seen in our interactions. Because of the fuzzy situation, the mixture of old, stable and very often implicit norms and new emerging norms, it didn't seem sensible to try to prove that certain norms were established in my data. Rather I am making interpretations of the production or negotiation of traditional or new norms. There is a very strong interpretative character in my analysis of norms. However, I will try to be exact and unambiguous in expressing my view of the status of each norm discussed.

The concepts of social and sociomathematical norms were developed by Cobb and Yackel (1996, 1998) to describe the classroom microculture. They wanted to understand and develop the processes taking place in whole-class discussions. I have in my data two small groups, and the discussions that took place in the peer interaction and teacher-students interaction in those groups. In my view the two small groups are embedded in the culture of the classroom in the teaching experiment. The students in the small-groups were members of the classroom community. All they had experienced was teaching and studying in traditional classrooms for years. On the other hand negotiation and production of norms in the two small groups was part of the negotiation and production of norms that took place in the class. I recognise that I am looking at the classroom microculture through the two small groups. It has to be acknowledged, though, that investigating and discussing in small-groups constitutes a different phase (Cobb, 2000) of a mathematics lesson than whole class discussion. I have tried to be explicit about which norms I have interpreted to be special for the small group discussions and which norms I see to refer to working in the mathematics classroom more generally.

2.5.4 Sociolinguistic subcultures of girls and boys

A related concept to social norms is the feature of sociolinguistic subcultures of girls and boys. In my teaching experiment the students, aged approximately 17, worked in groups of three or four and they were allowed to choose their partners by themselves. Almost all the groups consisted of girls or boys, which shows that in this context gender was an important factor for the students. While transcribing the discussions, I noticed the different styles of interaction of the girls and boys in the original four small-groups. I also noticed that the girls in one of the groups very often gave short bursts of laughters, which I interpreted as a sign of uncertainty. Maltz and Borker's (1982) concept of sociolinguistic subcultures seemed to describe well what I saw.

Daniel Maltz and Ruth Borker (1982) write in their classic paper about male-female miscommunication. Based on a wide range of research they argue that American women and men have differences in their conceptions of friendly conversation, by which they mean talk in informal, familiar settings. Children learn the rules for friendly conversation from peers at the age of 5 to 15, the time when boys and girls interact socially mostly with members of their own sex.

In their intimate and cooperative play in small groups or pairs, girls seem to develop friendships that involve closeness, equality, mutual commitment and loyalty. Malz and Borker argue that girls learn to do three things with words: 1) to create and maintain relationships of closeness and equality 2) to criticize others in acceptable ways and 3) to interpret, accurately, the speech of other girls. In order to maintain relationships of equality and closeness girls need to learn to give support, to recognise the speech rights of others, to let others speak and acknowledge what they say. In activities, they need to learn to create cooperation through speech. Girls also learn to criticize and argue with other girls without seeming overtly aggressive, without being thought to be “bossy” or “mean” (Maltz and Borker, 1982.)

On the other hand, Maltz and Borker describe how boys play in larger, more hierarchically organized groups. Important is the relative status. Hierarchies fluctuate over time and over situation. The social world of boys is one of posturing and counter posturing. According to Malz and Borker, boys use speech in three major ways: 1) to assert one’s position of dominance, 2) to attract and maintain audience and 3) to assert oneself when other speakers have the floor (Maltz and Borker, 1982.)

Maltz and Borker (1982) argue that "American men and women come from different sociolinguistic subcultures, having learnt to do different things with words in a conversation". Their theory represents the so called "cultural" approach to gender differences in language according to which the difference between masculine and feminine speech communities is thought to be stylistic. There is no predetermined preference for either style. But when both genders act according to their best understanding, there is a danger of miscommunication (Tannen, 1993a.)

I have written two papers (Partanen, 2005, 2007) where I have used Maltz and Borker’s theory for describing the style of interactions in my original four small-groups. In the first one of them, I analysed how the girls in the data expressed uncertainty through their talk when they were investigating mathematics, but I also remark that their style of talking may well be due to the way girls are supposed to talk in friendly conversations. In the second paper, I describe the different sociolinguistic subcultures of girls and boys in the four small-groups. My interest in using Maltz and Borker’s theory was not in testing the different cultures thesis (MacGeorge et al., 2004) or commenting on the origins of the different cultures, which is a debated question in feminist research (Cameron, 1996). Instead, I used the theory as a tool for describing what I had observed in the data.

The original four small-groups consisted of two groups of boys (three and four students) and two groups of girls (three and three students), one of which consisted of two girls and a male member, Veikko. My primary reason for determining the small groups for the video recordings was that there would be one group of girls and one group of boys with approximately similar prior achievement in mathematics from both of the two classes. The choice was difficult because of the great differences in prior achievement between the classes. The interaction in the group of girls of the open approach seemed originally to me rather girl-like, although there was the male member in the group. By analysing the discussions of the students, I actually have shown that Veikko could use in his talk both strategies typical of boys and those of girls. Thus to a certain extent this group of two girls and a boy can be seen to represent a group of girls. In the following, I am going to summarize the results from my two papers.

In the peer interaction of the four small-groups studied, the girls invited and encouraged others to speak and they acknowledged what the others said more than the boys. Maltz and Borker (1982) see these features as part of maintaining equal relationships and creating cooperation through speech. For example, the girls expressed proactive utterances which required (and received) a response and they used tag-questions. They also gave more positive minimal responses. The girls in the data gave more space for the others to express their ideas than boys. For example, they made propositions which were meant to enhance the mathematical discussion as questions or in conditional form. They also used hedges (Rowland, 2000), which enabled them to indicate their commitment to the truth of a proposition or attach vagueness to nouns, verbs or adjectives. These features of the girls' talk can be interpreted as trying to avoid giving the impression of mathematical authority and also recognizing the speech rights of others, which both contribute to building relationships of equality. Whether the girls really did listen to the messages of others in an equal manner is not clear on the basis of my analysis. I merely described the style of their talking. Some of the ways of talking of girls in the small-groups can also be interpreted as being expressions of uncertainty (Partanen, 2005, 2007.)

The boys in the four small groups were more assertive than the girls. They interrupted each other more often and they had disputes, boasting, name calling, jeering and mocking. They also gave more orders to each other than the girls. The boys seemed to be very often in the process of posturing and counter posturing, that Maltz and Borker (1982) refer to. Although these ways of talking of the boys can be seen as domination, I want to emphasize with Tannen (1993b), that boys also initiate and strengthen friendships through that kind of behaviour. Instead, I saw as domination the way that, especially, the most successful boys in my data made mathematical propositions as direct statements and orders. The way they talked conveyed mathematical authority and leadership, with no expectation that the others might have different ideas or wanted to express them (Partanen, 2005, 2007.)

Through their talk the girls and boys in my four small-groups established that it is not feminine, but masculine, to know mathematics and be confident in it. The way the girls talked is easily interpreted conveying lack of self confidence, and the way the successful boys expressed mathematical authority speaks for confidence in the domain (Partanen, 2005, 2007.)

2.6 Research on the use of small-groups in mathematics and science instruction

Many of the social norms produced and negotiated in my data must be seen as specific to the small-group phase of mathematics lessons. Thus, my project intersects with research on the use of small-groups in mathematics instruction. In the following I am going to discuss research results on that field mostly based on three research reviews, those of Good et al. (1992), Kumpulainen, (2002) and Bennet et al. (2010). Good et al. discuss grouping in mathematics instruction, Kumpulainen writes generally about teaching and learning via small-groups at school and Bennet et al. review recent research literature in science education.

Davidson (1985) criticizes the contemporary achievement comparisons (use of small-groups vs. traditional teaching) because they were made in reference to computational skills, simple concepts and application problems, only. He calls for more information about the development of high-order skills. Phelps and Damon (1989) argue that peer collaboration is a good method for promoting conceptual development, but not for enhancing rote learning. Noddings (1989) proposes two frameworks for supporting the use of small-groups in teaching; outcome and development frameworks. Developmental theorists focus on cognitive, social and moral development instead of just on learning the content of the traditional curriculum. Kumpulainen (2002) suggests on the basis of her review that collaborative learning in small-groups seems to offer students possibilities for developing critical thinking, skills in problem solving, interactional skills and innovative and reflective thinking.

Good et al. (1992) review research literature in the beginning of 1990's and state that most of the research done that far has been focusing on the learning outcomes of the use of small-group discussions. They call for programmatic research on small-group processes, both cognitive and social. Kumpulainen (2002) shows how research in the change of the millennium, indeed, reflects the attempt to describe and understand the processes in collaborative learning at school. For example Mercer (2000), Hogan, Nastasi and Pressley (2000) and Chain (2001) have shed light on the conditions and mechanisms of collaborative learning. Kaartinen and Kumpulainen (Kaartinen and Kumpulainen, 2001, 2002, Kumpulainen and Kaartinen, 2000, 2003, Kumpulainen and Mutanen, 1999) have studied both cognitive and interactional processes in small-groups in science and mathematics classrooms and Bjuland (2002, 2007) has investigated the characteristics of mathematically productive discourse of student teachers during collaborative problem solving in geometry. Hurme et al. (2006, 2009) have focused on analyzing metacognition in small groups solving mathematical problems in a computer network.

Collaborative learning in small-groups is a complex issue to be studied and it has been approached from many traditions and points of view (Good et al., 1992, Kumpulainen, 2002, Bennett et al., 2010). However, all the three research reviews conclude that the use of small-group discussions in instruction normally promotes students' mathematical learning and acquisition of high order skills, and that motivational issues may play a part in this. But it is not the use of small-groups per se which automatically brings forth progress. The approach must be properly implemented via "good instruction", for example by careful planning of organization and appropriate tasks (Good and Biddle, 1988, Noddings, 1989). It is also generally recognized that the quality of collaboration and interaction varies much from group to group, and that democratic and high quality interactions do not appear naturally in small-groups. Teachers and students need possibilities for developing their skills in communication and communal problem solving (Good et al. 1992, Kumpulainen, 2002, Bennett et al., 2010.) The rest of this section is devoted to discussing some important focus points in research on the use of small-groups in instruction.

Working in small-groups offers possibilities for students to be active in learning (Good et al. 1989 – 1990, Good et al., 1992, Kumpulainen, 2002). On the other hand many research reports show how involvement in small-groups has led to differentiated opportunities of participation for students as a function of achievement. Mulryan's unpublished doctoral dissertation from 1989 (Good et al., 1992) shows that high-achieving students manifested more quality attending behaviour. They have found to dominate small-group discussions because they are expected to be more competent (Cohen, 1982, Rosenholtz, 1985). They often direct task initiations and play leadership roles (King, 1989). In Webb's research (1989) help-giving further facilitated high-achievers because they had to clarify and organize their thinking when giving explanations to others. Low-achieving students in Cohen's (1982) and Rosenhoztz' (1985) studies were often relatively passive during group work because they were not expected to make contributions. Interestingly they, however, seem to enjoy small-group work (King, 1989). Low-achievers have been observed to have difficulties in understanding the group task (King, 1989) and more problems in comprehending what is expected student behaviour in collaborative small-group work (Good et al., 1992).

Also other factors, like gender for example, have been observed to be connected to the ways and possibilities of participating in collaborative small-group work. Lindow, Wilkinson and Peterson (1985) noticed that although second and third graders could handle controversy that arose during group discussions, the means for solving a conflict varied with achievement and gender. They further observed that boys and high achievers had more attempts of explaining one's position and had more answers accepted by other participants than girls and low-achieving students. Keys (1997), De Vries et al. (2002) and Tolmie and Howe (1993) identified clear differences in interactional styles in small-groups according to gender. All-male groups confronted differences in their individual predictions and explanations, whilst all-female groups searched for common features of their predictions and tried to avoid conflict. Mixed groups interacted in a more constrained way, and Tolmie and Howe (1993) thus suggest that the best of all-male and all-female group interactions was lost in them (Bennet et al., 2010.)

Esmonde (2009) compares two activity structures in secondary mathematics collaborative small-group work in terms of equity of participation. He shows how experts tended to dominate interactions during group quizzes, whereas group participation structure was more equitable in presentation preparations. His work shows, like that of Cohen (1994) and Jiménez-Aleixandre et al. (2000), that the type of tasks given to students has influence on the type of cognitive and social interactions that occur in the small-groups. Rohrkemper and Corno (1988) argue that teachers often want to enhance success and design tasks that are so predictable that they deprive student opportunity to explore alternative task- or self-regulation strategies. According to Cohen (1982), assignments which are typically given to individual students are the ones most unsuitable for small-group work. He continues that tasks that require multiple abilities and contributions for task completion are likely to promote collaboration by all students in a group. During the last two decades educational research has started to emphasize the suitability of open problems in collaborative small-group learning (Champange, 1992, Hogan et al., 2000, Kumpulainen and Kaartinen, 2000). For an open problem there is no one specific answer and the problem can be approached from many points of view. This has been recognized to enrich the collective negotiation of meaning in the small-groups (Kumpulainen, 2002).

There are no clear recommendations about whether small-groups should be heterogeneous or homogeneous. The decisions may depend on the situation of collaborative learning and the goals assigned for it (Cohen, 1994, Fernandez et al., 2001, Kumpulainen and Mutanen, 1999). Webb (1989) suggests on the basis of her review that mixed ability groups with high-, medium- and low-achievers are beneficial to relatively high- and low-achieving students but not to moderate achieving students. However, she concludes that two other forms appear to be beneficial to all students: mixed ability groups with high-ability and medium-ability students or medium-ability and low-ability students, as well as groups with all medium ability students (Webb, 1989.) Hogan (1999) found that friendship groups, which were generally single-sex, functioned more effectively and promoted better development of understanding than mixed or teacher-constituted groups.

Assigning certain roles for students to fulfil in group work is not recommended by Good et al. (1989 – 1990). In their research they found that sometimes the roles were artificial and in many cases students fought over the roles or ignored them. Richmond and Striley (1996) report that allocating roles has benefits when tasks are well-structured, but it is counterproductive in the case of poorly structured tasks adding to students' difficulties in engaging with the task. In line with Cohen (1994), Kumpulainen (2002) states that only when the goals for learning are simple like acquiring facts, understanding a certain text or mastering the routine use of certain concepts or strategies, it might be sensible to structure the collaboration by, for example, assigning roles for students. But Kumpulainen stresses that collaborative situations of learning which support high-level thinking and conceptual learning occur best in situations of open problem solving where it is not sensible to impose roles and models for working from outside. Some researchers have studied naturally occurring roles of students (Good et al., 1992, Kurth et al., 2002, Hogan, 1999, Esmonde, 2009). There are also studies which have investigated the nature of different collective strategies (discursive, cognitive or social) in small-groups which are based on mutual roles of the students (Cobb, 1995, Mercer, 2000, Kumpulainen and Mutanen, 1999, Esmonde, 2009).

The culture of the classroom and corresponding rules for interaction have an important role in developing quality collaboration in small-group discussions (Kumpulainen et al., 2001). Students' previous experiences in school, including their beliefs about appropriate and expected classroom behaviour, are likely to influence their behaviour in collaborative group settings. This is especially so when the small-group work is being introduced (Good et al., 1992.) Research has identified many problems in the working of small-groups, problems which I see to be related to classroom social and sociomathematical norms. Jiménez-Aleixandre et al. (2000) report that the small-groups in their research tended not to engage very often in processes which fostered meaning. Rather, they would reach agreement on the basis of finding something agreeable to all group members. And the conclusions were often reached by one or two group members exerting authority or by the "majority rule". Justification of claims was thus on a social basis rather than a result of a scientific argumentation. Newman (1990) notes that certain beliefs work against thoughtful inquiry in schools. Students and teachers tend to believe that knowledge is certain rather than problematic, it is created by outside authority, it is learned and expressed in small and fragmented chunks and it is learnt as quickly as possible. Several studies report students' low level of engagement with tasks (Bennett et al. 2010) and that the communication of students in communal problem solving is superficial and bound to the concrete level (Kumpulainen, 2002). Jiménez-Aleixandre et al. (2000) noticed that a large proportion of the talk in small groups was related to "doing the lesson" rather than talk about the intended focus of the talk. And, Newman (1990) reports that individuals sometimes have great difficulties in subjecting themselves to continuous scrutiny; and at times they are unwilling to resolve ambiguity and contradiction.

Many researchers suggest that training is needed for students and teachers in the skills required for handling and participating in group discussions (Hogan, 1999, Richmond and Striley, 1996, Zohar and Nemet, 2002). Richmond and Striley stress the importance of inclusive leadership and a climate of equitable participation. Hogan calls for guidance to the students by metacognitive training about the nature of collaborative learning, effective group strategies and awareness of what constitutes progress. This might help them in taking constructive roles in small-group discussions. Jiménez-Aleixandre et al. (2000) as well as Roth and Roychoudhury (1992) recommend coaching in argumentation skills for both teachers and students. Although the previous research reports don't mention the concept of norm, it seems to me that for researchers working in the emergent perspective these questions would be a matter of negotiating norms compatible with inquiry mathematics or inquiry learning in science.

Challenges for future research and developmental work on the use of small-groups in instruction are questions about how equitable participation could be fostered, how teachers and students could be supported in the development of efficient communication and collaboration skills, how affective factors are intertwined with the social and cognitive processes and the development of concepts and theoretical frameworks for analysing small-group activity (Good et al., 1992, Kumpulainen, 2002, Bennett et al., 2010.)

2.7 Negotiation of mathematical meaning and learning opportunities

In addition to investigating norms in my project classroom through the two small groups, I am interested in how acting according to the norms was connected to the occurrence of learning opportunities for the students. According to Blumer (1969) the position of symbolic interactionism is that in interacting with each other, individuals have to take account, or to interpret, what the others are doing. A person's actions are formed as she changes, abandons, retains or revises her plans based on the actions of others. In this way, social interaction is a process that forms human conduct, instead of just being an arena where human conduct takes place (Yackel, 2001.) Joint action (Blumer, 1966) grows from the participating persons' activity. Blumer stresses the collective nature of joint action:

> A joint action, while made up of diverse component acts that enter into its formation, is different from any one of them and from their mere aggregation. The joint action has a distinctive character in its own right, a character that lies in the articulation or linkage as apart from what may be articulated or linked. Thus, the joint action may be identified as such and may be spoken of and handled without having to break it down into the separate acts that comprise it. (p. 17)

Blumer, however, reminds us that it is important to see that "the joint action of the collectivity is an interlinkage of the separate acts of the participants" (p. 17). It is constantly formed, and each instance of it has to be constantly formed (Blumer, 1969.) The meanings and interpretations that underline joint action are continually subject to challenge. Thus both the individual actions and the joint action can change over time (Yackel, 2001.)

According to Blumer (1969) human beings act toward things on the basis of the meanings that things have for them. Such things include anything that the human being may note: physical objects, other human beings, institutions, guiding ideas etc. In the symbolic interactionism, meaning is seen to arise in the process of interaction between people. The meaning of a thing, for a person, grows out of the ways in which other persons act toward the person with regard to the thing. Meaning is thus a social product that is formed in and through the defining activities of people as they interact. But although meanings are formed in the context of social interaction, the use of a meaning by an individual is not just an application of that. Instead, the use of a meaning in action involves an interpretive process. In this process the actor indicates to herself the things toward which she is acting. The making of such indications is an internalized social process in that the actor is interacting with herself. Then because of this process of communicating with herself, interpretation becomes a matter of handling meanings. The actor selects, checks, suspends, regroups, and transforms the meanings in the light of the situation in which she is placed and the direction of her action. Thus, interpretation is not a mere application of established meanings but a formative process in which meanings are used and revised as instruments for the guidance and formation of action (Blumer, 1969.)

Bauersfeld, Krummheuer and Voigt (1988) applied sociological concepts from symbolic interactionism and ethnomethodology in order to deal with teaching and learning mathematics. In this research, I am applying the concept of *meaning* from Voigt (1994, 1996) in connection to the ideas of Blumer (1969). Voigt discusses negotiation of meaning in a first grade classroom. According to him, the focus of attention is on the process of mathematization, which is the transformation of empirical situations into mathematical statements and vice versa. When doing mathematics at higher levels, instead of empirical situations we can think about, for example, mathematical symbols, mathematical tasks and questions, mathematical concepts and mathematical procedures. I call mathematical objects anything that can be mathematized. For example, in the beginning of our first derivative investigation we typed the sequence of characters (f(z) – f(1))/(z – 1) | z = 4 into the symbolic calculators of the students after which they started working in small-groups. The expression (f(z) – f(1))/(z – 1) can be seen as a mathematical object. It can be transformed to a mathematical statement. It is the gradient of a chord to the graph of the function f(x) from x-value 1 to x-value z. The whole sequence in the previous example can also be seen as another mathematical object. It refers to the substitution of the variable z by four and thus to the gradient of the chord from 1 to 4.

There is ambiguity in the classroom, or in the small-groups, about how mathematical objects are interpreted. For example, in the first derivative investigation in group A Veikko had obviously been following the whole-class discussion at the beginning of the session. He interpreted the expression (f(z) – f(1))/(z – 1) to be the gradient of chord. He was talking about the steepness of the chords when z was made bigger and bigger.

55 Veikko: Siis mitä enemmän pistetään näitä, niin sitä enemmän tämä jyrkistyy näin.
The more we put these, the steeper it is.

56 Anni: Mutta miten tästä saadaan [epäselvää]?
But how do we get [indistinct]?

57 Veikko: Siitä tulee jossakin vaiheessa semmonen yheksänkymmenen asteen kulma. (Jenni näppäilee laskintaan)
At some point it will be a ninety degrees angle. (Jenni taps her calculator)

58 Anni: Ym.
Um.

59 Veikko: (painokkaasti) Vai tuleeko?
(with emphasis) Or will it?

At that moment Anni did not share the meaning of the expression with Veikko. But when listening to Veikko (55, 57) and trying to make sense of what he said, she made an interpretation which helped her to understand Veikko's utterances.

60 Anni: Onko tuo se kulmakerroin sitte tuo (osoittaa laskimeen)?
Is that the gradient, then (points to the calculator)?

61 Veikko: On. Pistetään näin, että x on kaks [epäselvää] jotenki ... (Anni ja Veikko ryhtyvät kirjoittamaan. Jenni katsoo mitä nämä tekevät ja tulee mukaan.)
Yes. Let's put that x is two [indistinct] some way … (Anni and Veikko start writing. Jenni is watching them and then she joins the activity.)

From then on, the students acted as if the expression were the gradient of a chord. But when I came to visit the group, I had yet another interpretation.

7 Opettaja: Eli te ootte ottanu niinkö ykkösestä ykköseen, sitte ykkösestä kakkoseen, niinkö? ... Ykkösestä kolmoseen. Niitä kulmakertoimia? (Anni ja Veikko katsovat opettajaan vähän ihmetellen. Jenni katsoo Annia.)
Well, you have evaluated them from one to one, then from one to two, right? … From one to three, those gradients? (Anni and Veikko look astonished. Jenni is watching Anni.)

8 Veikko: Niin. Eikö?
Yes. Haven't we?

9 Anni: Ym (nyökkää).
Um. (nods)

10 Veikko: Eikö me menty niin?
That's what we did, isn't it?

11	Anni:	Niin. *Yes.*
12	Opettaja:	Joo eli tuota tuossa, te ootte teillä on täällä vaan z:a. *Okay. So here you have, you just have this z.*
13	Veikko:	Niin ja me ollaan aina vaan vaihdettu sen arvoa, sen z:n arvoa. *And every time we have just changed its value, the value of z.*

I obviously interpreted the expression to be the gradient of a chord from x-value 1 to x-value z. The expressions of surprise suggest that the idea of all the chords starting from x = 1 was a new idea for the students. This view is supported by following the discussions of the students after I left the group.

Ambiguity in interpretations of mathematical objects may lead to *negotiation of mathematical meaning*, explicit or implicit negotiation. Negotiation can be seen as mutual adaptation to each other's actions and interpretations, as Blumer (1969) explains in his concept joint action. In the previous extracts we can see both explicit and implicit negotiation.

In my analysis, I am looking at the accomplishment of a given assignment in a small group as joint action. Woods (1992) stresses that symbolic interactionism is not simply consensual, but can cover all modes of interaction also confrontation, indifference and conflict. In the joint action of the small-groups, the negotiation of mathematical meaning and construction of individual interpretations takes place.

During the second extract, Anni and Veikko arrive at an agreement that the expression is the gradient of a chord and after the incident they act accordingly. In their negotiation they established a meaning taken-as-shared (Voigt, 1994). Voigt emphasizes that what is meant by meaning taken-as-shared emerges during the process of negotiation. It is a construct on the level of interaction i.e. sociological construct, not a cognitive element. It has its own right at the level of interaction the same way as the joint action can be seen to possess a character of its own which can not be reduced to the individual acts composing it. In my analysis I consider the negotiation of mathematical meaning and resulting meanings taken-as-shared as *the collective process of meaning construction*. The collective process of meaning construction is on the level of interaction, it is an observer's view of the process. It is an important part of the situation where the students in the small-groups do their individual interpretations and thus construct their *personal meanings*. Personal meaning is a psychological construct referring to conceptions and beliefs of individuals.

What is typical of constructivist approaches is to emphasize that the accomplishment of intersubjective meanings does not imply that the individual subjects "share knowledge" (Voigt, 1998). Only meanings taken-as-shared can be produced. People only interact as if they interpreted the mathematical topic of their discourse the same way. For example, if I had been following the discussion of Anni and Veikko during turns 55 - 61 where the meaning taken-as-shared was established, I would have agreed with them and I would not have noticed that they did not see that the gradient starts from x value 1, which was obvious to me.

Each person constructs knowledge from her experiences in processes that include the two ways distinguished by Bauersfled (1988): adaptations to the resistance to “the world” and negotiations in social interaction. There can never be a match between two persons’ conceptualizations, but a fit of conceptualizations means that the differences don’t disturb communication between them too much.

In my analysis I am applying Cobb and Yackel’s (1996) view of learning in locally social situations. In the small groups investigating mathematics the local situation for constructing personal mathematical meanings includes the collective process of meaning construction, the participation structure and the sociomathematical norms expressed and acted by the participants. The situation constitutes the conditions for the possibility of learning of the participants. According to Cobb (1995), when students mutually adapt to each other’s activity and attempt to achieve a consensual domain for mathematical communication, learning opportunities arise for them. By a *learning opportunity* I mean the opportunity for constructing new personal mathematical meanings that are compatible with the syllabus. Learning is, however, not a direct consequence of participation in the collective meaning construction. Individual students may differ in their interpretations. It is the individual process of interpretation and construction of mathematical meaning which results in actual learning. During my analysis I developed criteria for deciding whether, in a particular situation, a learning opportunity occurred for a student or not.

I think that analysing the occurrence of learning opportunities, in addition to just looking at learning, may enhance our opportunities for getting relevant information about productive learning situations.

2.8 Research questions

The research questions have evolved during my PhD studies and the research process. My experiences from the experimental course, my teacher’s experience and my theoretical reflections as well as my acquaintance with the data have contributed to their development. I felt that by analysing social and sociomathematical norms I could gain understandings of important processes in a situation when the investigative small-group approach was introduced. Because of the short length of the teaching experiment and because there was no intentional negotiation of norms, I could not suppose that new norms were established in the data. For that reason I have focused on interpreting the production and negotiation of norms in the data, both explicit and implicit.

I am going to approach the analysis of social norms through searching for regularities and interesting occurrences in the episode participation structures (Erickson, 1992). Based on my findings I am going to make interpretations of social norms being produced or negotiated. I am going to investigate all the expressions of mathematical beliefs and values in my data and make interpretations of sociomathematical norms produced or negotiated. As a teacher, in addition to describing norms in my data, I was also interested in the connections of norms to the learning of the students, more precisely to the occurrence of learning opportunities for the students in the small-groups. The emergent perspective together with additional ideas from the interactionist tradition helped me to formulate my final research questions and construct my analysis for answering the questions.

The research questions are the following:

1. What kind of social norms were produced or negotiated in the interactions of the two small-groups?
   - 1.1 What kind of participation structures did the students and the teacher establish in the two small-groups?
   - 1.2 What kind of social norms were produced or negotiated through the participation structures?

2. What kind of sociomathematical norms were produced or negotiated in the interactions of the small-groups?

3. How were the participation structures and acting according to certain sociomathematical norms intertwined with the occurrence of learning opportunities?

3 CONSTRUCTIVIST TEACHER RESEARCH ON INTERACTION AND LEARNING

3.1 Introduction

Cobb and Yackel (1996, Cobb, 1994) emphasize that their approach to development of theory is pragmatic. They don't discuss in detail the philosophical suppositions behind the emergent perspective. I went on reading about their background theories, especially about symbolic interactionism, interactional theory of learning and teaching mathematics and radical constructivism. In the following two sections 3.2 and 3.3 I am going to present some ideas about the premises of those theories and build a philosophy for my own research that would not, at least in a great extent, contradict those theories. In the section 3.4 I shall discuss the teacher research aspects and ethnographic character of my project. In chapter 3.5 the situation of the experimental course, and in section 3.6 the method of analysis, will be presented.

3.2 Ontology

Alan Bryman (2004) sees two main movements in interpretivism. One is the hermeneutic-phenomenological tradition and the other, more debated, is the theoretical tradition of symbolic interactionism. According to Bryman the general tendency in social science research has been to emphasize the similarties of the movements and to broadly see symbolic interactionsm as interpretive in its approach. This is especially so due to the writings of Herbert Blumer. Bryman, however, warns us not to exaggerate the parallelism between the two traditions. They both have in common an antipathy for positivism and in general an interpretative stance. But the hermeneutic-phenomenological tradition is a general epistemological approach, while the symbolic interactionism, at least the movement connected with Blumer, is a type of social theory that has "distinctive epistemological implications" (Bryman, 2004, p. 15.)

When writing about methodological principles of empirical science Blumer (1969) explains his stance. He partly agrees with the traditional position of idealism which says that the "reality" exists only in human experience. He writes: "I think that this position is incontestable. It is impossible to cite a single instance of a characterization of the "world of reality" that is not cast in the form of human imaginary" (p. 21). Blumer continues, however, that this position does not shift reality from the empirical world to the realm of the imaginary. The reality doesn't exist for human beings only in their minds. Such a position would make empirical science impossible. Blumer stresses that "t(T)he position is untenable because of the fact that the empirical world can "talk back" to our pictures of it or assertions about it - talk back in the sense of challenging and resisting, or not bending to, our images or conceptions of it" (p. 21). According to Blumer, empirical science is fundamentally an enterprise that seeks to develop images and conceptions that can successfully handle and accommodate the resistance offered by the empirical world under study (Blumer, 1969.)

Blumer (1969, p. 23), however, criticizes traditional realism. According to him, the obdurate character of the empirical world is not fixed or immutable in some ultimate form. Quite contrary, the reality of the empirical world appears "here and now" and is continuously recast with the achievement of new discoveries. He also criticizes the idea that the reality of the empirical world can be seen and described by the methods of physical sciences (Blumer, 1969.) It seems to me that the stance of Blumer described here might be close to what Heikkinen et al. (2005) call combination of ontological realism and epistemological constructivism. Another possibility for interpretation of these writings of Blumer is that he is speaking about a perspective in empirical social science, and thus he is only explaining his epistemological stance, and not at all speaking about ontological issues.

In their presentation of the interactional theory of learning and teaching mathematics, Bauersfeld, Krummheuer and Voigt (1988) distance themselves from the views of Blumer. They disclaim the possibility of the existence of all kinds of objective facts. "So-called objective realities are accessible only via subjective interpretations as constructed across social interaction. So-called objective truth or rightness herefore cannot transmit themselves through evidence. Rather they are formed or subjectively realized through active interpretations and via social processes of communication and negotiation" (p. 175). According to Bauersfeld and colleagues the theory is valid for an observer as well. The observer has access to meanings, structurings and norms of validity only through interpreting reconstructions, which is a concept from Cicourell (1973). But the reconstructions are interpretations of an already interpreted reality rather than images or interpretations of objective data (Bauersfeld, Krummheuer and Voigt, 1988.)

The conceptions of Bauersfeld and colleagues are compatible with the view of reality in radical constructivism. This theory emphasizes that human beings construct their knowledge from their experiences. Instead of a match with ontological reality, radical constructivism talks about the viability of a subject's constructions. Our knowledge is viable if it stands up to experience and enables us to make predictions and bring about or avoid certain phenomena (von Glasersfeld, 1995a.) Radical constructivism is a psychological and epistemological theory and it does not, specifically, discuss the question whether there exists an external reality or not.

It seems to me that both radical constructivism by von Glasersfeld and the interactional theory of learning and teaching mathematics of Bauersfeld et al. emphasize that we can *know* about the "so-called" reality only by constructing knowledge from our experiences. But the theories don't take a clear stance on whether there exists an external reality, or on the nature of that reality. Therefore, in this research I see that the most important reality is in the human mind and that the ontology of the research is relativist. This means that there exist several, local and specific realities (Lincoln and Guba, 2000) which are constructed through experience and social interaction.

3.3 Epistemology

Bauersfeld, Krummheuer and Voigt (1988) refer to the different theoretical perspectives behind their interactional theory and state that they all have a certain constructivist perspective. The essence of the social constructivism represented by the emergent perspective is the coordination of interactional theory with radical constructivism. In line with these perspectives I see that my epistemology is constructivist (Lincoln and Guba, 2000). As a researcher I am making my interpretations about the meanings the students and I, as a teacher, gave to phenomena and about the social interaction in my data. In the research report, I am presenting my reconstructions and by doing that producing, or creating, reality. My writings represent a specific version of the phenomena. The knowledge produced must be seen to be indeterminate and it must be subject to continuous critical reflection and revision.

An issue, then, is the researcher's presence in the interpretive text. I have my subjectivity and identity, which are formed in social processes (Berger and Luckman, 1966). I have my personal history and my history of being categorized by other people. One cannot separate these characteristics from the research process of interpreting and meaning construction. The knowledge produced in this research must be seen as constructed by the female mathematics teacher of the students with an additional role of a research student.

According to Berger and Luckman (1966) scientific institutions are also socially constructed. The activities of people make up and constitute institutions, which through routines and historization begin to seem objective to the individuals. Thus the norms of science, its ways of understanding knowledge, reality and truth are historically constructed (Berger and Luckman, 1966.) This view opens the opportunity for criticism of science and for doing science differently.

I am not only recognizing the researcher's presence in the research process. I am consciously allowing my subjectivity to enter the research act (Harding, 1986, Kincheloe 1991). I am not treating it as a bias; instead, I am trying to make use of it in the interpretation and analysis. What is important, however, is that the knower and the status of knowledge should be explicitly reflected. Anna Sfard (2001) has similar ideas about assessing what she calls effectiveness of communication:

> First we must always keep in mind that it is an *interpretive* concept: any assessment of communication is based on personal interpretations of the discourse. The speaker compares her intentions to the effects her statement had on an interlocutor; an observer - a passive participant - compares the intentions evoked *in him* by the different interlocutors he is watching and listening to. Different participants - and this includes the observer - may have differing opinions on the effectiveness of the same conversation. Thus, when it comes to the evaluation of communicative efforts, it is important to be explicit about whose perspective is being considered. (p. 49)

In his methodology for critical constructivist teacher research, Kincheloe (1991) refers to feminist epistemologies and encourages teacher researchers to take this direction: "O(o)ur notion of critical teacher research maintains that inquirers connect knower and known, purpose and technique by utilizing the human as instrument" (p. 31). When reading, watching and listening to the discussions and actions of my students and myself, and trying to make sense of them, I am utilizing my personality as a female and as a mathematics teacher. For example, as an upper secondary high level mathematics teacher, I have a certain sensitivity for observing the mathematical thinking of students at that level. It would be much harder for me to try to go into the world of first graders and make interpretations of their mathematical constructions.

In my research project, I am also analyzing my own actions. I am looking at myself as a teacher participant in the interactional processes of the classroom. I am claiming that as a female teacher researcher in my late 40's, with my personal dispositions, I can take a specific view of the teacher, different from if the teacher or the researcher was somebody else. I don't have to worry about all the ethical issues. I can have at the same time a critical and an empathetic attitude towards the teacher. At first, recognizing my failures as a teacher was not easy. But soon I discovered that what was deficiency for me as a teacher might be a strength for me as a researcher. And, on the other hand, exposing and analyzing my own actions gives me a great opportunity for developing my personality and the use of the investigative approach in my own teaching.

Utilizing our subjectivity in the research process, however, does not mean that any unreflective subjective account will do for scientific knowledge. A good tool in sharpening the approach is reflection (Ronkainen, 1999.) For example, during the early phases of the analysis, through reflection, I consciously made a change in my attitude concerning the mathematical constructions of the students. As a teacher, especially during the experimental course, it was important to me whether the students had constructed the mathematical ideas that were intended by me. But later, the interest of just understanding the thinking of my students became a more prominent goal. And, of course, the feasibility of the knowledge claims a researcher makes must be justified, and argued in relation to other possible claims (Ronkainen, 1999).

3.4 Methodology

It is not possible to situate my project exactly and label it as a representative of one specific type of methodology. In addition to aiming at the construction of knowledge in the field of mathematics education, my research also has practical goals. Through my project, I am developing my own practice and, I hope, my study will have relevance for other teachers at the upper secondary level applying and developing the use of the investigative small-group approach. My project is teacher research (Cochran-Smith and Lytle, 1999, Altrichter et al., 2008) and it shares some features with action research (Carr and Kemmis, 1986, Elliot, 1991, Kincheloe, 1991). In my approach there is a general ethnographic orientation and, furthermore, I have applied some ingredients from ethnographic microanalysis of interaction (Erickson, 1992).

3.4.1 Teacher research, contributions to theory and practice

Barbara Jaworski (2004) states a simple but an important consideration that in mathematics education research we have two fundamental research aims, firstly to enhance knowledge, and secondly to enhance practice in the field. The traditional way of thinking about the two aims is that from knowledge produced by research, implications are deduced for practice. It has, however, been recognized that not so much influence has occurred in this direction. An important question is how we go beyond learning from the outcomes of research to using the research process as a developmental tool. Developmental research projects try to respond to this challenge by including teachers in the research act (Jaworski, 2004.) I see that this can happen in two ways; teachers could be employed in collaborative projects by researchers, or they could do inquiry into their own practice.

The teaching experiment of my thesis was an attempt to apply an innovation in my own practice, and I am at the same time the teacher and the researcher in my study. My PhD project is part of the teacher research movement (Stenhouse,1975, Elliot, 1988, Cochran-Smith and Lytle, 1999, Altrichter et al., 2008). Teacher research at present stems from many traditions and it has been applied in various ways (Cochran-Smith and Lytle, 1999). Very often it is, however, conceptualized as action research (Carr and Kemmis, 1986, Elliot, 1991, Kincheloe, 1991, McNiff and Whitehead, 2006, Altrichter et al., 2008). My project shares many ideas with the different traditions, especially with action research, but I haven't been following faithfully any one of them. In the following, I am going to comment on the teacher research aspects of my research.

There is a close connection between practice and theory in my research. The research questions arose from my experiences during the experimental course and while transcribing student discussion. In the analysis, I used a theory that seemed to fit my practice and gave me helpful concepts by which to gain understanding of the situation in my experiment. While analysing my data, I had a very special opportunity for doing reflection-on-action (Schön, 1983), which according to action research philosophy (Altrichter et al., 2008) is an important feature of professional action. I feel that I have both developed and grown as a teacher. These results definitely feed back into my practice. And perhaps they give me new or refined research questions for future projects even though the spiral of cycles of Lewin and Elliot (Elliot, 1991) for action research is not built into my research design. On the other hand, in line with Jaworski's (2004) challenge, my research project also attempts to contribute to the development of theory in the field.

All the teacher research movements construct the role of the teacher as knower and agent both in the classroom and in larger educational contexts (Cochran-Smith and Lytle, 1999). By the late 1950's, the general tendency in educational research separated science and practice (Kincheloe, 1991, Carr and Kemmis, 1986). Simultaneously the expansion of the cult of the expert happened. Others, than teachers, had the right to define what is knowledge about education. The neo-positivism, and even the 21st century "new orthodoxy" (Hodkinson, 2004), have contributed to these ideas still persisting in our society. From the social change tradition of teacher research, which draws on critical theories, Kincheloe (1991) argues that "t(T)his separation of knower and known, this epistemological distancing, produces a tacit logic of domination between researcher-researched and knower and known" (p. 38). The objectified and abstracted knowledge handed down to teachers by some educational researchers makes a vast difference between the experiences of teachers. Teachers should take research in their hands, so that they become "active producers of meanings - not simply consumers". At the same time our view of research should be changed (Kincheloe, 1991.) Through my research and my development to a researcher, I have gained knowledge, attitudes and self confidence to cross the gap between researchers and practitioners. And even more, I have started appreciating the advantages of the standpoint of a teacher while doing educational research. My double identity of a teacher researcher helps me in interacting and working efficiently in both of the contexts, at school and in the world of research.

Denzin and Lincoln (2000) recognize that qualitative research can be flexible and that the blurring of genres is taking place. For example, Lincoln and Guba (2000) define themselves to be social constructionists, but their ideas for doing research also reflect strongly the tendency to action embodied in critical theorists' perspectives. I also see in my project a shade of critical constructivist teacher research (Kincheloe, 1991). In addition to my own empowerment, the intention in my experimental course was that more students would have access to important mathematical ideas about the basics of calculus. My opinion, informed by research and practice, is that, in Finnish upper secondary high level courses, only the high achievers have the opportunity for constructing appropriate conceptual understanding of the basic concepts of calculus. In the contemporary syllabus, four of the following courses (30 % of the total time devoted to studying mathematics in upper secondary high level) were based on these concepts. Equity in mathematics education can be defined as access to mathematical ideas that have clout (Bruner, 1986), as access to further mathematics courses and as an opportunity for developing interest and self efficacy in studying mathematics (Cobb and Hodge, 2007). In that sense the experimental course was an attempt to increase equity in my high level class. Another related point is that the goal of my research is to learn about social and sociomathematical norms in my mathematics classroom. This informs me in my future attempts to negotiate norms which increase student autonomy and democracy when students investigate mathematics in small-groups.

The scientific character of action research is sometimes questioned by referring to the lack of distance of the researcher from the situation of collecting data, to the (im)possibility of part time researchers to achieve the quality criteria of scientific research and by criticizing the situatedness of the knowledge produced (Altricther et al., 2008). However, at present action research is a well established qualitative research methodology with theoretical foundations. According to Altrichter et al. (2008), the aims of action researchers in the UK reveal that it is not only a model for further development of professional practitioners but also a contribution to theory. Furthermore, the opportunities for producing knowledge by action research have advantages that traditional science may lack; close connection of knowledge and practice and thus its relevance to practitioners. I had the opportunity of adding distance to my experiment and data as well as to myself at that time because developing the analysis took so long (many years) and because of the two years that I could take off school teaching and concentrate on my research work. My study is a PhD-project so I have to take seriously the quality criteria of doing scientific research. And there exist different ideas for the generalization of knowledge drawn from cases in qualitative research methodologies.

3.4.2 Ethnographic orientation

Typical of my research is a certain ethnographic orientation (Spindler and Spindler, 1992, Tedlock, 2000, Hammersley and Atkinson, 1983). I began the experimental courses with the intention of establishing a quasi experimental study for comparing two different styles of the investigative approach. Just in case my plans would change, I collected different kinds of data during the teaching experiment. When teaching the courses and beginning transcription, I saw two themes emerging to me from the data. I revised my plans and decided to focus on these topics. My research questions and my whole study have evolved all the time with respect to my observations, experiences, theoretical reflections and the opportunities of my data. Spindler and Spindler (1992) describe the evolving character of the ethnographic study, which is in line with my way of working.

The aim of my research is to describe the cultural climate in the experimental class through analyzing the social and sociomathematical norms present in the interactions of two small-groups. An additional aim is to sketch how acting according to the norms was intertwined with the occurrence of learning opportunities for the students. I have adopted theoretical constructs from the emergent perspective and applied them in describing what I as a teacher felt important phenomena when beginning to use the investigative small-group approach. So my results can be seen as theoretical description (Hammersley, 1992).

Ethnography involves close and prolonged interaction with people in their everyday lives (Tedlock, 2000). When my teaching experiment took place, I had been teaching in Lyseonpuiston lukio for six years. I am also a former student of the school. Before the experiment, arrangements were made to let me become acquainted with the students. I was the mathematics teacher in many of their previous courses and I was the pastoral form teacher for one of the classes. There was a big difference between my status as a teacher and the status of the students. I felt, however, that learning mathematics was the area in my students' lives that I naturally had access to because of my being their mathematics teacher.

My main data are the video recordings of the discussions in two groups for six sessions. My project shares with microethnography or ethnographic analysis of interaction (Erickson, 1992) its type of data and its view of putting immediate interactions into a broader context. In microethnography, audiovisual recordings of authentic face-to-face interactions are analyzed in detail, with greater precision than it is possible to do in participant observation. The ethnographic interest in combining levels or aspects of social organization leads the researcher not only to attend to information on the screen but also to the information that comes from beyond the screen, from wider participant observation and from social research more generally. "Ethnographic microanalysis portrays immediate human interactions as the collective activity of individuals in institutionalized relationships who, as they enact daily life locally in recurrent ways, are both reproducing and transforming their own histories and that of the larger society within which they live" (Erickson, 1992, p. 223.) According to these ideas I used the other types of data, my acquaintance with the students, and my knowledge of the situations as background information when interpreting the meaning of our actions and interactions. When doing this I tried to follow Erickson's (1992) instructions that background information should only be used for behavioral evidence from the recordings.

Although my method bears some resemblance with the procedures of microethnography (Erickson, 1992), I am not following them intentionally or precisely.

3.5 Empirical research

3.5.1 Setting

Lyseonpuiston lukio is a Finnish upper secondary school of approximately 700 students in the capital of Lapland, Rovaniemi. Around 60 000 inhabitants live in the city and its surroundings. The school was founded a hundred years ago, and it was the first school in our city which led to the matriculation examinations and to Finnish universities. During the experimental course, there were two other, but smaller, upper secondary schools in the town. The school buildings of Lyseonpuiston lukio are situated in the centre of Rovaniemi with good bus connections, and because of that the school attracts students from the surroundings of Rovaniemi. For a long time, Lyseonpuiston lukio had been the "top school" in the city, but in the 80's and 90's its status was somewhat normalized. However, the long traditions continued influencing the culture of the school to some extent.

Students normally attend Finnish upper secondary school for three years, but personal schedules may be planned for two to four years. At the end of their studies the students take part in national matriculation examinations in at least four, but normally five to six subjects.

The school year consists of five periods, 7 and half weeks each, and the periods end with an examination week which lasts a week and two days. The subject matter is divided into courses so that one course (approximately 25 hours including the three hour examination) is studied during one period. Students normally choose 4 to 7 courses per period from the choice of all the courses in the school. The courses are divided into compulsory and optional courses. For example, at the time of the teaching experiment, there were 10 compulsory courses and 3 optional courses in high level mathematics. In addition, school specific courses may be offered. At the time of the experimental course there were three such courses in our syllabus for mathematics. The system of courses means that each class or group studies together for 6 weeks only. A student may have many different teachers in one subject during the school year. There were 5 teachers with a major in mathematics in our school. When needed, some physics or chemistry teachers also taught high level mathematics courses.

3.5.2 The experimental course

The experimental course was one of the compulsory courses for high level mathematics in Finnish upper secondary syllabus, the basics of calculus. For the compulsory and optional courses there is a normative national syllabus. Because more time was needed for the investigative approach and the pre-test, two topics were translated from the course to the next compulsory course in the whole school. The post test was at the same time the course examination. One of the school specific courses at that time was a revision course in calculus. It was planned that if there were problems due to the experimental course for some students, they would be recommended to attend the revision course. The students had very little prior experience of other than traditional mathematics teaching.

The second year students, approximately 17 years old, worked in friendship groups of three to four. Almost all of the groups were single-sex groups. Originally I had two classes (27 and 31 students), and for video recordings I chose one group of girls and one of boys from each class so that the groups were as similar as possible. The two classes were very different in their prior achievement. This may be caused by a technical reason in the choice of courses. I chose the groups so that their members were approximately average achievers. But the decision was difficult, and because I looked more to the achievement than the gender aspect, the group of girls in the open approach class included a male member. My decision was also influenced by the observation that the interaction in that group seemed rather girl-like to me in spite of the male member. Finally, I have excluded the class with the structured approach from the data, and I have focused in my analyses to the two small-groups in the class of the open approach (31 students). They worked with questions as open as I thought was possible.

3.5.3 The two small-groups

My main data includes the discussions in the two small-groups of the open approach for six sessions. Group A consists of two girls Anni and Jenni and a male member, Veikko. The names of the students are changed. The previous achievement of the girls was somewhat above average. Their mean marks from the first six courses in high level mathematics were 7.7 and 8.3 (the best mark is 10 and the smallest accepted mark is 5). Veikko's corresponding mean from 5 previous courses was 6.4 which is a little below the average. In the pre-test, there were questions planned to measure conceptual understanding about functions, their change and rate of change as well as velocity and acceleration. Anni's performance was the best in the group. Jenni left many questions unanswered, and she wrote later in her diary that she believed that she had not succeeded very well. Veikko gave a few nice answers, but on the other hand he showed some serious problems. For example, very often he confused change in function values and change in x-values.

At the beginning of the course, Anni and Jenni were happy that a mathematics course would be taught in a different way than normally, but Anni was worried about how she would learn to use the complicated new calculator. Veikko was enthusiastic about the new fine calculator, but at the same time he also claimed that it was not so nice to start learning to use new technology when he just had gained confidence with his own graphics calculator. But, in his diary, he said that he had decided to try to be positive. Veikko had spent the previous year abroad as an exchange student. I don't know in which country. He wrote in his diary that during that year he had studied very easy mathematics and that he was a little bit nervous about whether he would remember all the stuff he had studied earlier in our school. His comments hint that he might have been attending some calculus classes during the previous year.

Group B was made up of four boys: Mika, Juha, Pekka and Reijo. Mika's previous achievement was somewhat above average, his mean from the first six courses was 8.3. Reijo's success was average with mean 7. Juha and Pekka had performed below the average, their corresponding means were 6.2 and 5.8. Mika performed best of the students in the pre-test. Juha also showed rather good conceptual understanding. But contrary to the order of marks of Juha and Reijo, the latter did not succeed very well in answering the questions. Pekka was like Veikko in the other group. He gave some sophisticated answers. For example he was the only one of the students in both of the groups to identify negative change for function values. But he left many questions unanswered.

At the beginning of the course, Mika and Juha wrote in their diaries that the idea of working in small groups was a promising one and that the beginning of the course had been easy for them. Reijo wrote only about the mathematics being studied, and Pekka never returned his diary. Mika and Pekka were brothers, and all the boys knew each other. For example, during one lesson the boys discussed Juha's visit to Pekka and Mika's home. The boys had common hobbies; they spent a lot of time with computer games and on the internet and some of them also seemed to be engaged in designing computer games.

Generally speaking, the students in the two small-groups had been average achievers in their previous studies of high level mathematics in the upper secondary school. At the beginning of their first year, all of them, except Veikko, had written a short text about their own relationship to mathematics. In those writings they show self confidence and positive attitudes. When the experimental course was about to begin the students expressed that they liked the idea of working in small groups.

3.5.4 Instructional design

Mathematics instruction in the upper secondary high level classes of my school is normally done through teaching from the front. First, the homework from the previous lesson is revised so that certain students present their solutions on the blackboard and the teacher comments on the solutions and revises connected ideas. Then, the teacher presents the theory of the next topic through a combination of a lecture and a discussion. Usually students interact with the teacher only on an individual basis; they are not supposed to comment to each other. The lesson continues so that important examples are presented by the teacher after which there is possibly time for practicing the exercises in the textbook. For homework students are recommended to read a few pages from the textbook and to solve certain tasks. The experiences of students about mathematics lessons from lower secondary and upper elementary classes are very similar, I think.

The investigative small-group approach

In my experimental course, we used the investigative small-group approach to study different limits and the concept of a derivative. Before teaching a topic, I asked the students to discuss in small-groups certain questions or to solve some problems. All the students had the symbolic calculator TI-92 in their use during the course. The students were instructed to keep records of their working, and almost every time I asked them to write a summary about their findings. The aim was to give them experiences of the concepts/methods to be studied in concrete situations and, if possible, to make them construct important features of these by themselves. I did not consciously use any specific learning theory in constructing the questions. Rather, I made use of my interpretations about the investigative approach and my teacher's experience. During the small-group phase of the lesson, I walked around the class, discussed with the students, responded to their questions and challenged them with my questions. The video recordings of the two small groups took place in separate rooms near the classroom. I didn't have good enough equipment for the recordings to be done in the classroom.

After the small-group sessions we came together. Sometimes we began this whole-class phase by reviewing the summaries of all the small-groups, but often time was scare and only one summary was presented. Then I tried, through teaching from the front, to connect the theory of mathematics to be learnt to the experiences of the students. This meant symbolizing and mathematizing the concrete examples and situations, as well as reflecting the ideas developed by the students. It has to be noted, though, that I was not very skilled in orchestrating the whole class discussions.

Many groups failed in finishing the investigation about the concept of a derivative, problem 4. The students in those groups felt frustrated and started showing irrelevant behaviour. I had to interrupt the working of the small-groups and start the whole-class session, even though some other groups were showing promising progress. But even then the students were already in the concrete situation through which I could explain the idea of instantaneous velocity as a limit of average velocities and as a gradient of tangent.

In the following, I am going to present the assignments given to the small-groups, and describe how they were connected to the whole course. I will clarify my goals for the students' working. For the first assignment, I shall also describe roughly the actual solution processes in the two small-groups.

Problem 1

Problem: What is the limit of the function f(x) = $\frac{\sin x}{x}$ when x approaches zero?

- Construct such an infinite sequence of x-values, the numbers of which are greater than zero and approach zero.
- Construct an infinite sequence of x-values, the numbers of which are smaller than zero and approach zero.
- What happens to the values of the function f(x) = when x approaches zero from the right? What about when x approaches zero from the left?
- Sketch the graph of the function.
- Write down a summary of your findings.
- * Investigate the limits of the functions $\sin\left(\frac{1}{x^2}\right)$ and $x\sin\left(\frac{1}{x^2}\right)$ at x = 0.

Before presenting problem 1 to the students, I had shown them examples of the concept of an infinite number sequence.

Through the two first tasks, I wanted the students to get an experience of how we think that the x-values change in the process of finding a limit for a function at a particular x-value. The “epsilon-delta” definition is not required of the students in upper secondary schools in Finland. Instead, a process type of conception of a limit is sufficient for them. Another aim for these two tasks was that the students would construct a meaning for the word “approach”, meaning, such that we can make the x-values as close to zero as we ever wish. When answering the third question they might see the similarity with the previous questions, now only the values of the function are approaching the number 1 as close as we ever wish. I believed that constructing the required sequences through discussions would demand deeper and more versatile intellectual processing than just accepting a given sequence and considering what happens to the x-values in it. I also believed that this deeper processing would end in stronger experiences and personal meanings constructed by the students.

By asking the third question, I wanted the students to get an experience of what it means that there is a limit for the function at x = 0. At a certain stage the calculator was not able to show the deviation of the function values from one. The students got the result 1.0. To help them to construct the limiting process, I equipped them with a sheet made by a mathematical computer program of the function values for very small positive x-values. The function values were given to so many decimal places that was enough to show that in each case the function value was still less than 1, but when x approached zero the function values were all the time closer and closer to one. That the question is separately for x values approaching zero from the right and from the left, I think, might help the students to construct ideas of the limits from the right and from the left.

I asked the students to sketch the graph of the function because I wanted them to construct connections between the numerical observations and the graph of the function. Through sketching the graph, the students might notice that the function is not defined at x = 0. This observation would help in separating the concepts of the value of the function and the limit of the function.

The aim of the summary was to make the students recall and reflect the previously mentioned ideas. The last question was planned for advanced students to deepen their ideas if they had to wait for the others.

The solution process in group A

Led by Veikko, the group first discusses that the function $\frac{\sin x}{x}$ is not defined at x = 0. They write down the observation and start discussing the third question. On the basis of the graph, Veikko and Anni together construct that when x approaches zero from the right or from the left, the function values increase. I visit the group and help them to begin constructing the required sequences, which the students then finish. During the previous lesson we had been studying the increasing and decreasing of functions. When discussing the third question again, the students decide, against their previous comments that x always has to increase and they write their answer so that first the function values increase and then decrease.

I come to visit the group. Veikko asks how they should think about the changing of x-values. I explain the difference between the way x-values change when we investigate increasing and decreasing of functions and when we investigate the limit of a function. The students return to their conclusion that when x approaches zero from the right or from the left the function values increase. I urge them to do some numerical work and express it more precisely. I ask the students to describe how the function values increase.

I bring to the group the sheet presenting function values for very small positive x values approaching zero. The students discuss a long time about the trends in the digits of the approximate values. Then Anni gets an idea: The function values get all the time closer to one. In their summary the group first writes only about the changing of x-values. But, after my prompting, they continue by describing that the function values approach one but never get there.

The solution process in group B

Juha was absent from the session.

The boys begin by searching information from the textbook and the formula booklet. They then try to construct a formula as an answer to the first question. Mika understands that they could also work in concrete terms and constructs a sequence; one divided by ten, one divided by hundred, ... Reijo and Pekka accept the solution. Together they write down the two required sequences.

By skilfully using the calculator and investigating the properties of the graph of the function sinx/x, the boys, led by Mika, construct that there is no function value when x = 0 and that when x approaches zero from the right or the left, the values of the function approach one but never get there.

I urge the students to do some numerical work as well. Pekka constructs x-values 3, 2 ½, 2, 1 ½, 1, 1/2. Mika demands that the x-values should get much closer to zero so that one could see the function values approaching one. Mika claims to me that the calculator does not give the right function values, because very soon he got the value 1.0. I change the number of digits that Mika's calculator is using and give them the sheet with function values for very small positive x-values. Soon Mika declares that his calculations prove their earlier hypothesis. Reijo agrees with him.

Pekka finds from the textbook a statement for the same limit and reads it aloud: "The values of the function seem to approach number one when x approaches number zero". Reijo reads a more general description of the limit of the same function at x = 0. In that it says that x is not allowed to be zero. The boys have a fierce discussion about what it means that one is a limit for the function. Pekka describes it: "If one is the limit, it kind of touches a little bit, touches a little bit, touches a little bit, but is not quite there." Pekka alone has written the statement he had read from the book as a summary. I notice it and tell the group that they have a nice summary, there. The other boys look astonished but copy the summary from Pekka's notebook. It says: The values of the function seem to approach number 1 when x approaches number 0.

After the first small-group session I taught the formal definition of the concept of limit (the process type of definition). We found more limits numerically. Then I gave the students some rules for finding limits symbolically, which we then practised.

Problem 2

- What happens to the values of the function $f(x) = -\frac{x}{(x+1)^2}$ when x approaches number -1?
- What happens to the values of the function $f(x) = -\frac{x}{x+1}$ when x approaches number -1?
- *How could we predict the difference from the expressions of the functions?

At the time of the second small-group session, I assumed that it was a mathematical practice in the classroom to interpret the meaning of the word "approach" so that something gets as close as we ever wish to a certain number. Through the first question, I wanted to give a concrete experience for the students about what it means if we say that the limit of a function at a certain x-value is infinity. And through the second question, I wanted to give an experience that sometimes the limits from the right and the left may be “different types of infinities”. The last question, an extension, was intended to offer an opotunity for constructing connections between graphical/numerical processes and the symbolic representation of the function.

After the second small-group session I taught how we deal with questions involving “limits infinity and minus infinity” in the examinations. We did some exercises.

Problem 3

Problem: What do the limits of a function at infinity and at minus infinity mean?

- Construct an infinite sequence of x-values, the numbers of which increase above all boundaries.
- Construct an infinite sequence of x-values, the numbers of which decrease below all boundaries.
- What happens to the values of the function $f(x) = \frac{1}{x}$ when x increases or decreases without boundaries?
- What happens to the values of the function $f(x) = \frac{x^2}{x^2 - 1}$ when x increases or decreases without boundaries?
- What happens to the values of the function $f(x) = x^3 + 1$ when x increases or decreases without boundaries?
- Write a summary.
- * Investigate when a rational function has a limit zero at infinity, when the limit is a number different than zero and when the function values increase or decrease without limits.

Through the two first questions in problem 3, I wanted to give an experience to the students about what it means that x-values “go to infinity or to minus infinity”. Again, they had to construct the sequences by themselves because I believed it would contribute to stronger experiences and personal meanings. Through the following three questions, I wanted to give them an experience about a limit at infinity and about the “limit” infinity at infinity. By the summary, I wanted them to reflect on the previous ideas and construct a conception of the limit at infinity, if it was not done earlier. The last question, again, is an extension through which the students would organize the examples above and other possible examples generated, and construct connections between the graphical/numerical and symbolic representations.

After the third small-group session, I taught the students the definition of the limit at infinity and minus infinity and gave examples about finding those limits symbolically. We spent some time practicing corresponding tasks.

After working with limits, the concept of continuity of a function was taught by traditional methods. I felt that I wanted to save time for the investigations concerning the concept of a derivative.

Problem 4

Problem: A glider is moving on an air track. Find its velocity at 1 s from the start.

- Construct a function which describes the distance of the glider from its initial position as a function of time.

- Draw with your calculator chords to the graph starting from x = 1.

- What does the gradient of a chord mean?

- What is the instantaneous velocity of the glider at 1 s?

- What is the graphical meaning of the instantaneous velocity?

- Write a summary.

Before students work with problem 4, we solved the first question together. I measured certain time-distance pairs for a glider on an air track in front of the class. Initially the glider was at rest, and its movement had a constant acceleration. Together we used the modelling properties of the symbolic calculator to find a quadratic function of best fit. I advised the students to leave out the first order and constant terms of the function. We drew the result function for the same picture with the data, and saw that the function with the quadratic term only fitted very well with the data. I did all this with the students because we were late in the schedule.

The questions here are arranged as they were presented to the students. I wanted to give freedom for them to choose the order in which they would approach the total problem of finding the instantaneous velocity of the glider at one second.

My goals for this investigation were that the students would construct that, in this context, the gradient of chord is the average velocity of the glider, and that the gradient of the tangent is its instantaneous velocity. I further intended that the students would construct the limiting process inherent in this, that when the time interval is made shorter and shorter, the average velocities approach the instantaneous velocity, and gradients of the chords approach the gradient of the tangent.

Many groups could not finish this investigation while some groups had already done everything required and others were just about arriving to interesting results. Because of the task-off behaviour of some students, I had to stop the small-group session and start the whole-class discussion. The summary of one group was presented, after which I taught the concept of the derivative in the context of our situation.

During the following lessons, we applied the concept of the derivative in finding average and instantaneous rates of change for different functions. I taught the concept of the derivative function and we studied how to differentiate polynomial functions. Our approach to the derivative function was investigative by accident. In the end of a lesson we derived together the gradient of tangent at x for the function $f(x) = x^2$. The time was out and I gave as homework for the students to consider what can be done with the result. At the beginning of the following lesson some students showed very sophisticated ideas about the derivative function and its connections to the graph of the original function and it was easy to build my teaching on those ideas.

Problem 5

Problem: How do we construct the equations of a tangent and a normal to a curve at a given x-value?

- Find the equation of a tangent to the curve $y = \frac{1}{3}x^3 + x^2 + 1$ at $x = -1$.
- Find the equation of the normal to the same curve at the same point.
- Write a summary.

During the previous period, the students had studied analytic geometry. The formula for finding the equation of a line was used in that course. Also the formula for the relationship between the gradients of two perpendicular lines was studied. My goals for this investigation were that the students would use and connect their prior knowledge and construct the methods for finding the equations of a tangent and a normal to the graph of the given function at a certain x-value. Through writing the summary, they would reflect on the methods and become more aware of them.

After this small-group session, I taught the students how to solve typical problems in examinations by the methods they had constructed. As homework, the students did corresponding exercises.

Problem 6

Problem: How can we make conclusions about the increasing and decreasing of a function from its derivative function?

- Draw with your calculator tangents to the graph of the function f(x) = $\frac{1}{3}x^3 + x^2 + 1$ at different points. How are the tangents connected to the increasing and decreasing of the function?
- How is the derivative connected to the increasing and decreasing of a function?
- The graph below is the derivative function g´(x) of another function g(x). Sketch in your notebook the graph of the function g(x).

- Write a summary.

My aims for this sixth investigation were that the students would construct that when the derivative function has positive values, then the function itself is an increasing function and vice versa. I also wanted them to construct more thoroughly the connections between the graphs of the function and its derivative function. I wanted the students to see that the value of the derivative function seen as the y-coordinate on its graph is the gradient of the tangent to the graph of the original function.

After the reflection on the results of the small-group session I quickly taught the students how to solve examination questions which ask the students to investigate the increasing and decreasing of a function. The method required of the students involves construction of a certain type of a system of signs which can easily be connected to the knowledge obtained through the investigation.

Analysis of the assignments

The assignments used in the investigative small-group approach reflect my understandings and interpretations of the British tradition of investigations (Morgan, 1997) and how these could be applied in Finnish upper secondary mathematics. The approach is different from the realistic mathematics education (Gravemeijer, 1997, Gravemeijer and Doorman, 1999) which Cobb et al. have used in their teaching experiments.

In realistic mathematics education, students are given sequences of everyday problems for which there are multiple solution methods. Individual students' solutions are discussed, compared, justified and their efficiency is evaluated. The aim is that formal mathematics evolves from students' own solution processes. The role of the teacher is to select the instructional activities, to initiate and guide discussions and to reformulate selected aspects of students' mathematical activity (Gravemeijer, 1997, Gravemeijer and Doorman, 1999.) Gravemeijer (1997) characterizes the method as a bottom-up approach for a good reason. In my teaching experiment, during the small-group sessions, students also constructed informal situated knowledge about the mathematical concepts and methods, but I attempted very soon to try to connect the experiences and constructions of the students to the formal mathematics to be studied. I guess that Gravemeijer (1997) would classify my method as a top-down approach, because the mathematical concepts and methods were taken as given and the attempt was only to try to connect the informal situated knowledge of the students with this pre existing system. But if we compare the investigative small-group approach to traditional teaching, there is a huge difference. I think that the nature of the negotiation between the two types of knowledge matters. If the students in my approach are active in the process of negotiation and the teacher utilizes their ideas wisely, there might be seen traces of an intermediate approach. I am not suggesting that I was such a skilled teacher during my teaching experiment, but I think that I have developed a little in that direction after the experiment.

When comparing the collective solution processes of problem 1 in the two groups, one can see that they were very different. And the students did not proceed in the order and the way that I had planned or imagined. The openness of the investigations gives room for students' creativity. But the actual processes also give a hint about the clash of the new way of studying with the old norms. Students in group A tried to follow my instructions from the previous lesson instead of trusting their ability to construct new knowledge. The boys in group B searched and tried to apply knowledge from the textbook.

Students' experiences and opinions of the investigative small-group approach varied to a great extent. I shall present here the comments from two students which represent the extreme cases. Anni, from group A, wrote in her diary at the end of the experimental course: "Really a nice way of teaching a course. You have internalized the subject matter much better when you have had to understand it through problems. This is the direction all mathematics courses should be developed to! I hope that in the future different styles of teaching are tried because the traditional lecture style in mathematics is the worst of all. I was so happy about this course that if I had to give a mark to the teacher it would definitely be a full 10." But, another student in the class criticized the course in his strong personal style: "I hated this course. First of all, students should be informed before hand that the course is not taught in a normal way. This kind of applying "one's own didactics" sucks. Lesson work with the sheets did not teach me anything, and the three lessons during which we practiced exercises in a normal way just confirmed the rule. In the exam it was "nice" to solve tasks, when you didn't have developed any routines. For some others this kind of approach may perhaps work, but I got only a hole in my knowledge about mathematics. If this kind of courses were offered more often, students should be informed about that." Later, this student admitted that his bad experiences may reflect the quality of interactions in his small-group. He was advised to participate in the school specific revision course of calculus, which he also did. Some students wrote in their diaries that they had found the questions in the investigations very good and helpful.

3.5.5 Data

My primary data includes video recordings of the small-group discussions of the two groups for six investigative sessions. My visits to the groups and my discussions with the students are included in the data. The recording of the first part of the derivative investigation, problem 4, in group B failed. The microphone was not on. The latter part of the investigation on limits at infinity and minus infinity, problem 3, in group A is also excluded from the data. Another mathematics teacher came to the room and started testing distance teaching facilities, and the discussions of the students were obviously interrupted and influenced by the incident.

As secondary data, I have video recordings of the introductions and finishing of the lessons, pre-and post tests, learning diaries of the students and my own diaries. According to the principles of ethnography, I have also used my acquaintance with the students and the school culture to interpret occurrences in the primary data.

At the end of the experimental course I asked the students in the two small-groups to evaluate how working in front of the camera had influenced their behaviour. Generally they thought that the influence of the camera was not too strong. In group A the students wrote that they had not expressed themselves totally freely. And in group B, three of the four students said that video recording made them discuss more about the mathematical topic than normally. From the tapes, I could conclude that the students were conscious about the camera. Sometimes they sent me greetings through the tapes and sometimes they were joking about the possibility of solving an argument of theirs by playing the tape. But there was a big difference between the interactions of the students when they were working as a group in front of the camera and when I was actually present in the group. The situation of working in a small-group when the camera is on is a specific type of situation of its own, which is different from both working without a camera and from my being present in the group. I am, however, convinced that the presence of the particular human beings in the group was more essential than the knowledge about the camera, especially towards the end of the course.

The results from the post test and the learning diaries were used for assessing the students' performance in the compulsory mathematics course. But, before the pre-test, I had told the students that the scores obtained from these would not influence the marks they would get from the course.

3.6 Method of analysis

I devised my own application of a method, originally developed by Paul Cobb (1995) for analysing interaction and conceptual learning in small-groups of second graders. Only later, I noticed some similarities with my method of analysis and the ethnographic microanalysis of interaction (Erickson, 1992). I first transcribed all the discussions in the small-groups. I wrote down all the utterances as well as special gestures and occurrences which attracted attention. If a person started speaking simultaneously with another speaker, I have shown the moment where the overlapping began.

I divided the transcribed discussions into episodes according to the theme of the discussion. This phase of my analysis bears some resemblance with the idea in ethnographic microanalysis of interaction (Erickson, 1992) of identifying the major constituent parts of an event or distinguishing sub segments of those. The discussions between me and the groups of students constitute separate episodes.

3.6.1 Analysing the episodes

The episodes were first analysed in chronological order. In addition to reading the transcript, I also watched the corresponding video clip. The two modes supported each other in helping me to construct my interpretations of the actions of the participants in the small groups. The analysis of each episode included inferring:

1) Episode participation structure
2) The mathematical beliefs and values expressed or acted in the group
3) The collective process of meaning construction at the small group level
4) The learning opportunities that arose for each student
5) The actual learning of the students, i.e. the construction of personal mathematical meanings made by each student.

To analyse the episode participation structure, I made a check list to consider, for each episode, aspects of participation in communication, participation in working together and the roles of the students and the teacher in knowing and learning. I also tried to be open to aspects that were not on my list. But the list helped me in the huge mental effort of analysing thoroughly all the 140 episodes. Analysing the social participation structure is also one aspect of the ethnographic microanalysis of interaction (Erickson, 1992). But Erickson suggests that only segments of special interest should be analysed more thoroughly. With Erickson (1992), I stress that the emphasis on analysing episode participation structures is on the dialectical and ecological relationships of mutual influence among participants in the event, not on the actions of individual persons considered in isolation from the actions of others.

I had another check list for analysing the mathematical beliefs and values expressed by the participants in the small groups. That list was not as useful as the previous one, and again I was looking also for aspects that were not anticipated by me.

To describe the collective process of meaning construction, I tried to act as an observer and look at the group as a whole. I tried to capture the collective nature of the joint action.

Beginning to analyse the learning opportunities that occurred to the students in each episode, I had to have a certain idea of the previous learning and personal meanings of each student. Here, the practice of analysing the episodes in chronological order was very important. I used the information available from my interpretations of the previous episodes, my knowledge of the syllabus and my acquaintance with the students. I have adopted the concept of learning opportunity from Cobb's (1995) analyses. It is not clear to me whether it is for him a theoretical or a practical one. On the one hand it is theory-laden in that the learning theoretical commitments of the researcher influence the decisions (Cobb, 1995). On the other hand, in his analyses Cobb (1995) sometimes goes to the concrete situations and deduces from the actual learning that a learning opportunity had occurred. I want to be explicit about my way of using the concept. In line with the emergent perspective, I developed criteria for deciding whether a learning opportunity occurred in an episode for a student or not.

First of all, some new meanings for the student must occur in the collective process of meaning construction. I decided to take account of only those new meanings which are in line with the conceptions of the mathematical community. If the student constructs the new meaning by herself, there is a learning opportunity. When a student actively participates in the process of meaning construction together with other students, there is a learning opportunity for her. I included into the latter category the cases where the student had actively participated in the discussions in the near past and now seemed to be listening to the persons making the construction of a new meaning. I decided that a learning opportunity did not occur when a student was not following the discussion in the group, when a student was not listening or did not hear the persons constructing the new meaning. I did not consider learning opportunities to occur when two students were just expressing their own thinking and did not seem by any means to adapt their expressions to the speaking of the others. I decided that there were no learning opportunities for the students when they clearly seemed to remain unsure about their conclusions. There were a few situations where I could not make up my mind. For example, in an episode in group A I adopted a strategy called Topaz effect (Brousseau, 1984, 1997, Novotna and Hospesova, 2007). Instead of developing students' ideas, I imposed my own version of the phenomenon. By using simple questions, I led the group into a conclusion which was not in line with the students' thinking. In the next episodes there were no signs of the students using the new meaning by themselves which we seemed to have arrived at together. In the few unclear situations, I went to see whether learning had occurred or not and this influenced my decisions.

I see that especially in the use of the concept of learning opportunity, my analysis coordinates sociological and psychological perspectives. To be able to make the decisions, I had to interpret the previous actual learning of the individual students and the meanings they ascribe to the mathematical objects at hand. But at the same time I have had to follow the collective process of meaning construction in the small groups.

After deciding that a learning opportunity occurred for a student a in an episode, I followed her actions in the subsequent episodes and tried to see whether the student had actually learned the new meanings and used them. These analyses then formed the basis for the further analyses in the following episodes. In some cases there was not enough information available for the interpretations. I am convinced that, in addition to searching for the actual learning of the students, it was fruitful to apply in my analysis the concept of a learning opportunity.

In the following, I am going to demonstrate, through a concrete example, my way of analysing the episodes.

The students in group B are working with the tangent and normal investigation (problem 5). They have differentiated the third order polynomial function and evaluated the gradient of the tangent by substituting x in the derivative function by -1. Mika has found from his notebook a formula $y - y_0 = k(x - x_0)$ and Juha has stated that to be able to use the formula, they need one point and the gradient. They boys have evaluated the y-value when x is -1 and obtained the result 5/3. Somehow, Mika has copied from his notebook the formula $y - y_0 = a(x - x_0)^2$. The boys had a fierce debate about whether there should be the second power in the formula or not. Other boys, except Juha, follow Mika who claims that in the formula $(x - x_0)$ has to be squared. When trying to apply his formula Mika starts doubting it. Episode 10 shows how the working of group B continues.

Session 10, Episode 5

253 Mika: Se ei oo ihan oikein kyllä mun mielestä tuo nytte.
I feel it's not quite right.

254 Pekka: y on miinus x
y equals minus x

255 Reijo: Onko se ihan oikein?
Is it right?

256 Mika: No ko miten se voi olla joku x toiseen juttu?
Well, how can it be something squared?

257 Reijo: Se vain nyt on. Lasku on matematiikkaa [epäselvää].
It just is. Calculating is mathematics [indistinct].

258 Mika: Ei sitä, ei se, miten tuo muka on noin?
We can't, it doesn't, how could it be like that?

259 Pekka: Miinus x toiseen miinus kaks x miinus kaks on
Minus x squared minus two x minus two is

260 Reijo: Miten niin väärin?
Why do you say it's wrong?

261 Mika: En mie tiiä. Ku siinä ei oo [epäselvää] x toi, tuo niinkö toiseen, muttako ei siinä kyllä oo.
I don't know. Coz it hasn't got [indistinct], the square, but actually it hasn't got.

262 Juha: Arvaa miksi?
Guess why!

263 Mika: No?
Well?

264 Juha: Sie oot laskenu paraabelin. Sillä voi laskea semmosen kaavan samalla lailla.
You've got a parabola. There is a similar formula for them.

265 Mika: Niin.
Yes.

266 Juha: Oisko siksi?
Could it be the reason?

267 Reijo: Ehkä, ehkä ei.
Perhaps, perhaps not.

268 Juha: Jätäppä se toiseen,
Leave out the square,

269 Mika: Se on, se on muute niin. Se on paraabeli tuo.
That's, that's right by the way. It is a parabola.

270 Juha: jätäppä toiseen
leave out the square

271 Reijo: Siis eikö siihen tuukkaan se toiseen?
So it isn't a square?

272 Mika: Ei.
No.

273 Juha: Jätäppäs toiseen siitä pois, niin katotaan se vastaus.
Leave out the square and let's see the answer.

274 Mika: Tuo, niin just, Juha, oisko se paraabeli tuo?
It, quite right Juha, could it be a parabola?

275 Juha: Mitä, kuulinko mie oikein? Juha on oikeassa.
What did I hear? Juha is right.

276 Reijo: Ei mutta oottekste vielä päässy, selvinny siinä ku se polynomi-juttu?
No but, have you found, have you agreed about the polynomial matter?

277 Pekka: Juha, [epäselvää] Juha on oikeassa.
Juha, [indistinct] Juha is right

278 Juha: (naurahtaa) Ei olla. En anna periksi. Yks muistikaavahomma.
(gives a short laugh) No we haven't. I am not giving up, the binomial formula.

279 Mika: No se on kyllä. Se oli selevä juttu se.
But I am quite sure. Nothing to be discussed.

280 Juha: Ei, eipä höpötetä. Sie käytit justiin äsken sitä meikän kaavaa. Älä sönkötä.
Don't, don't talk rubbish. You just used my formula. Don't babble.

281 Mika: Hä?
What?

282 Juha: Sie käytit sitä justiin äsken meikän tavalla sitä polynomifunktiota, niin älä sönkötä
You just used the formula in my way, so don't talk rubbish.

283 Mika: Missä kohasa tässä on polynomifunktio? (Pekka naurahtaa) Se, semmonen polynomifunktio
Where do you see a polynomial function? (Pekka gives a short laugh) Polynomial function

284 Juha: Siis ei polynom ku se muistikaavahomma..
No, I mean the formulae for binomials …

285 Mika: Niin, me ei puhuttu ees siitä muistikaavasta. Me puhuttiin siitä a miinus jutusta.
We were not talking about the formulae for binomials. We were talking about the, a minus thing.

286 Pekka: Niin.
Yes.

287 Reijo: Niin onko se nyt miinus x?
So is it minus x then?

288 Juha: Ai se suluissa a miinus b toiseen, a plus b toiseen?
Oh, you mean in parentheses a minus b squared a plus b squared?

289 Reijo: Miinus x miinus yks.
Minus x minus one.

290 Mika: Eikö a miinus b kertaa a plus b. Eikö me siitä puhuttu?
No, a minus b times a plus b. We were talking about that, weren't we?

291 Juha: Ei.
No.

292 Mika: Vaan siitä toisesta?
But about the other.

293 Reijo: Onko se nyt miinus x miinus yks?
Is it minus x minus one?

294 Juha: Just näistä muistikaavahommasta, siitä suluissa a plus b toiseen.
About the formulae for binomials, in parenthesis a plus b squared.

295 Reijo: Okei, mitä sit pitää tehä?
Okay, what next?

296 Mika: [epäselvää]
[indistinct]

297 Reijo: Hei, mitä tässä pitää tehdä.
Look, what do we need to do here?

298 Juha: Niin, elikkä silleen niinku mie selitin sen.
Well, like I was explaining.

299 Mika: No ei.
No, but.

300 Juha: Joo, joo. Tottakai. Mie oon oikeassa, aina.
Well, well. Of course. I am right, always.

301 Reijo: Onko tässä vielä jotaki kaavoja?
Do we still have some formulae here?

302 Mika: Siitä tullee miinus x
It becomes minus x

303 Juha: Plus kaks kolomasossaa, oisko?
Plus two thirds, could it be?

304 Mika: Niin.
Yes.

305 Pekka: Miinus x plus
Minus x plus

306 Mika: Kyllä kesti.
It took a long time!

307 Juha: Kylläpä kesti. Oisit voinu uskoa minua heti alussa. Oisitta saanu heti oikian sen. (opettaja saapuu paikalle ja tutkii Reijon vihkoa)
It took a long time. You could have believed me in the very beginning. You would have got it right straight away. (teacher joins the group and studies Reijo's notebook)

308 Mika: No ois voinu sanoa, että se oli
Well, you could have said that it was

309 Juha: Miehän sanoin, että se on paraabelin kaava.
I told you it's a formula for a parabola.

310 Mika ja Pekka: Ethän sie sanonu mittään. (yleistä hälyä)
No, you didn't say anything. (everybody speaking)

311 Juha: Sanoin monta kertaa.
I told you many times.

312 Mika: No et sanonu yhtikäs kertaa. Meillon videonauha, voiaan kattoa sitte illalla kuule.
No you didn't say even once. Listen, we've got a video tape there, we can watch it in the evening.

313 Pekka: Niin.
So, yes.

314 Juha: Mie sanoin, että se on paraabelin kaava. (erikoisella äänellä) Eikä ole.
I told you it's a formula for a parabola. (in a special voice) No it's not.

315 Mika: No et sanonu mittään.
But you didn't say anything.

316 Pekka: Et niin.
You didn't.

317 Mika: Kerran oot sanonu ääneen sen.
Once you have spoken out.

318 Pekka: Niin.
Yeah.

I wrote the results of the analysis of this episode as the following.

SESSION 5, EPISODE 10

EPISODE PARTICIPATION STRUCTURE

In the beginning of the episode (253 – 274), Mika is leading, Reijo and Pekka are following him, and Juha is waiting for the boys to notice their mistake. During the rest of the episode, the discussion in the group is closer to a democratic debate. Mika is mature enough to admit that Juha was right. Mathematical justification was for him more important than supporting his status. Juha celebrates loudly his being right (275, 300, 307). Reijo sees the tension and the possibility of a quarrel, and he transforms the topic of the discussion to be about an old, but more playful, debate of Juha and Mika. There are many phases in this episode when the students don't collaborate. They work in different pairs or alone. During turns 253 – 259, Pekka is doing some calculations and Juha is waiting for his moment to interject. Reijo is asking questions which remain unanswered, and Mika is thinking aloud by himself. During the debate of Juha and Mika (281 – 300), Pekka and Reijo go on working. Juha continues mocking the others and praising himself and the episode ends in a quarrel between Juha against Mika and Pekka (307 – 318). In the episode, Mika and Juha express their own mathematical thinking. The students listen to each other, but there are also problems with this respect. Juha interrupts the other boys three times (284, 294, 298), and all the others do it once (260, 269, 277). Many times Reijo is not able to make his questions heard (287 – 301). Mika did listen to Juha at the critical moment. There occurred two justifications (Mika 256 and Juha 264). Mika shows agreement (265, 269, 274, 304). Juha and Mika disagree with each other (Juha: 264, 273, 300, Mika: 283, 299).

MATHEMATICAL BELIEFS AND VALUES

Mathematical justification was more important to Mika than trying to support his status. He admitted having been wrong and that Juha was right, when Juha suggested that the formula he was using was the formula for constructing the equation of a parabola. Reijo expresses a belief (257) that in mathematics you don't necessarily need to understand, performing the right procedures is mathematics as well. Juha got his strength for resisting and challenging Mika from the knowledge that he had obtained the right answer. A few moments earlier, he had checked his answer on the calculator. Juha expressed a strong belief that the right answer is an indicator of the right method.

COLLECTIVE MEANING CONSTRUCTION IN THE SMALL-GROUP

Mika starts doubting that a second order polynomial can represent a tangent. Then Juha explains that the formula $y – y_0 = a(x – x_0)^2$ which the other boys have used is a formula for constructing the equation of a parabola. He continues by suggesting that when the second power is left out, the formula $y – y_0 = a(x – x_0)$ will give the equation of the tangent. Mika shows agreement with Juha in that what they had obtained was the equation of a parabola. The boys make the substitutions needed, and get the equation $y = -x + 2/3$, which answer they interpret as the equation of the tangent. Again, at the end of the episode, Juha announces that he had told the other boys it is the formula of a parabola. Mika and Pekka seem to agree with Juha about the mathematical matter. A meaning taken-as-shared seems to have been constructed.

LEARNING OPPORTUNITIES THAT AROSE FOR THE STUDENTS

All of the students had a learning opportunity for re-constructing or refreshing the meaning that the formula $y - y_0 = a(x - x_0)^2$ is for finding the equation of a parabola. Juha constructed the knowledge (during this episode or a little bit earlier). Mika had a personal problem which was solved by Juha's idea, and he participated actively in constituting the knowledge in the group. Pekka and Reijo had participated in using the formula and now they were listening to the discussion of Juha and Mika.

Mika, Reijo and Pekka got a learning opportunity to construct that the equation of a tangent is obtained by using the formula $y - y_0 = a(x - x_0)$. Mika was actively involved in the process when Juha expressed the idea; he accepted Juha's idea, applied the formula and also advised Reijo to do so (272). Most probably this was also a logical consequence of the first learning opportunity for Mika. Reijo was listening to Juha and Mika's discussion (271 – 272) and applied the knowledge. Pekka was also listening to the discussion and wrote the formula $y - y_0 = a(x - x_0)$ in his notebook.

ACTUAL LEARNING THAT THE STUDENTS DID

Mika did learn both of the ideas (EPISODE 18, 245) as well as Pekka (309, 310, EPISODE 18, 244). Juha learnt that the equation $y - y_0 = a(x - x_0)^2$ is the equation of a parabola. He constructed the knowledge. There is no data about Reijo's learning.

3.6.2 Episode-by-episode analysis

Originally, it was my intention to continue my analysis of participation structures as Cobb (1995) describes he had done in his research. In the second phase of the analysis, the inferences and conjectures made in the first phase became data that were meta-analyzed to develop chronologies of the children's social relationships and mathematical activity and learning (Cobb, 1995). I did so too, but I started developing the descriptions of typical participation structures already during the analysis of the episodes. After each episode, I made conjectures about the typical interactions in the group, and when I had a look at the following episodes I revised and developed the conjectures. After analyzing all the episodes the work continued as in Cobb's analysis. The idea of comparing instances across the research corpus is also included in the ethnographic microanalysis of interaction (Erickson, 1992).

In the way I described, a holistic picture developed in my mind and my writings, which I, then, tried to present in the first results chapter. I continued revising my conjectures and writing until I felt a certain satisfaction. It was a feeling that now the description was as complete as it was possible for me to construct. To support some of my claims, I also calculated frequencies of certain types of utterances. After describing the typical participation structure and some interesting, but not so typical, features in it for one group, I concluded what kind of social norms were being produced or negotiated through them. To a certain extent, in my results chapters, I have done what Erickson (1992) calls recomposing the smaller fragments of events into bigger wholes.

The further work with mathematical beliefs and values was simpler. After finishing the analysis of the episodes for one group, I classified the cases. Then, I used my knowledge about the culture of our school and about the students to interpret whether an old norm had been acted or a new norm was being negotiated.

The final part of my analysis included looking at the occurrence of learning opportunities. With each learning opportunity I described what kind of interactional features discussed in the previous analyses were contributing to the occurrence of that learning opportunity. And if an obvious learning opportunity was for some reason destroyed, I also described the reasons for that. I then classified the interactional features and their way of contributing to the occurrence of learning opportunities. My method of analysis did not capture the relationship between the occurrence of learning opportunities and nature of mathematical talk; whether the talk is about instructions or mathematical objects. That would have required a deeper approach and a research project of its own.

I hope, I have not exaggerated the similarities of my method of analysis and the procedures recommended for ethnographic microanalysis of interaction (Erickson, 1992). From the different methodologies, this tradition seems to be the nearest to my method.

4 PARTICIPATION STRUCTURES AND SOCIAL NORMS

The first research question was divided into two sub questions:

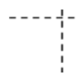
1. What kind of social norms were produced or negotiated in the interactions of the two small-groups?
   1.1 What kind of participation structures did the students and the teacher establish in the two small-groups?
   1.2 What kind of social norms were produced or negotiated through the participation structures?

In this chapter I am presenting the answers to these questions. I shall first focus on the communication and cooperation in the two small-groups during the investigations. I am going to describe typical and interesting features of the participation structures, both in peer interaction and teacher-students interaction. Based on these descriptions I am going to draw conclusions about social norms being produced or negotiated through the interactions. I shall then look at the participation structures in terms of knowing and learning, again separately in peer interaction and teacher-students interaction of the small-groups, and make inferences about the production or negotiation of corresponding social norms.

4.1 Participation structures and social norms in communication and co-operation

4.1.1 Participation structure in group A

Part of an episode from the third small-group session illustrates many features of the typical participation structure in this group. I had planned the assignment, problem 3, to be an introduction to limits at infinity and minus infinity.



Graph 1 The graph of the function $f(x) = \frac{x^2}{x^2-1}$

Episode 1

The students in this group had constructed the sequence 1, 2, 3, 4, … , and obviously another sequence, the terms of which decrease as required. They had discussed already the behavior of the first function, and now they were beginning to study the second function $f(x) = \frac{x^2}{x^2-1}$ (graph 1).

79 Anni: (osoittaa kynällään kysymyspaperia) Sillonku x kasvaa rajatta, niin ne, mitä ne sillon tekkee?
(points to the question paper with her pen) When x increases without boundaries, then they, what do they do then?

80 Veikko: Äh, kasvaa rajatta. (naurahtaa)
Um, increase without boundaries. (short laugh)

81 Anni: Ei mut siis
No, but

82 Veikko: [epäselvää] (Anni ja Veikko naurahtavat)
[indistinct] (Anni and Veikko give a short laugh)

83 Veikko: No niin, mitä siihen pitäs kirjottaa? [epäselvää] (Anni naurahtaa)
Well, then, what should we write there? [indistinct] (Anni laughs a little bit)

84 Anni: Et sillonku se x kasvaa rajatta, niin mitä tuolle funktiolle tapahtuu?
That when x increases without boundaries, what happens to that function then?

85 Veikko: [epäselvää]
[indistinct]

86 Veikko: Tuo on tuommonen peilikuva ittestänsä, eiköookki. Siis niinku näin yleisesti, muuten vaan. (Anni naurahtaa)
It's a mirror image of itself, isn't it? Well, generally, it seems. (Anni gives a short laugh)

87 Veikko: Siis mie ymmärrän, että tuossa on tuommonen äärettömyystarkotus mutta se ei vaan niinku näy tarpeeksi pitkälle.
Well, I understand that there is such an infinity purpose, but we just can't see it far enough.

88 Anni: Se niinku pienenee koko ajan vaikka sitä, siis ku mennään sinne oikealle, vaikka sitä ei nää ku se pienenee niin hirveän vähän.
It in a way it decreases all the time although, I mean if we go to the right, although we can't see it because it decreases so incredibly little.

89 Veikko: Saavuttaakohan se ikinä sitä nollaa siellä? (kaikki tutkivat laskimiaan)
I wonder whether it ever reaches zero there? (all the students study the screens of their calculators)

90 Jenni: Se eka kasvaa siihen miinus ykköseen ja sitte se, sitte sitä ei oo määritelty siinä. Ykkösestä eteenpäin se pienenee. Miten siinä välissä? (katsoo Annia)
It first increases to the minus one and then, then it is not defined at that. What about there between? (looks at Anni)

91 Veikko: Voikohan näitä tietää ees, että miten isoja arvoja se saa?
Can we even know how big values it gets?

92 Anni: Ym.
Um.

93 Anni: Sie voit laittaa sieltä table setistä sen.
You can change in the table set the

94 Veikko: Niinhän se oli.
That was it.

95 Anni: Mie oon seittemässä kymmenessä kahessa menossa. (naurahtaa)
I am at seventy two already (gives a short laugh)

96 Veikko: Seitten kaks
Seventy two.

97 Anni: Täälä se, kyllä se pienenee koko ajan.
Here it, yes it decreases all the time.

98 Veikko: Ei, mutta ei se ikinä saavuta sitä, ykköstä pienemmäksi mee.
No, but it never reaches it, it never goes smaller than one.

99 Anni: Mennee se ku se [epäselvää].
It will, when it [indistinct].

100 Veikko: [epäselvää] (Anni naurahtaa)
[indistinct] (Anni gives a short laugh)

101 Anni: Tai, en mie oikeastaan tiiä. Mie oon yks pilkku nolla, nolla, nolla ykkösessä. (naurahtaa) Ei se pienene enempää.
Or, actually I am not sure. I am at one point zero, zero, zero one. (gives a short laugh) It's not gonna decrease further.

102 Veikko: Mitä?
What?

103 Anni: Ei se pienene enempää.
No, it's not decreasing further.

104 Veikko: Niin se ei mee niinkö sen ykkösen alle. Sen takia ko tuosson tuo miinus ykkönen.
So it is not going below that one. It's because there is this minus one.

105 Anni: Ei, mennee se. Oota. En mie tiiä vielä. Se jää siihen ykköseen.
No, it will go. Wait. I don't know yet. It remains at that one.

106 Veikko: Jääkö?
Does it?

107 Anni: Jää. Ko se on, esimerkiksi viiessäsaassa niin se on vielä yks. Miten ois tuhat? Joo. Se on ykkönen.
Yes. Because for example at five hundred it is still one. How about a thousand? Yes. It's one.

108 Veikko: Vai niin. [epäeselvää]
Oh yeah?

109 Anni: Miten
How

110 Veikko: Eihän ne saavuta sitä, ehkä. [epäselvää] (naurahtaa)
They are not going to reach it, perhaps. [indistinct] (laughs a bit)

111 Anni: Ym. Ym.
Um. Um.

112 Veikko: Hyvä ku huomasit. (Annikin naurahtaa) Miten se siis muotoillaan, muotoillaan nyt sanallisesti?
Well done that you noticed. (also Anni laughs a bit) How can we put it in words, then?

113 Anni: Sillonku se lähenee tuota miinus ykköstä tuolta negatiiviselta puolelta, niin se kasvaa. Mihin tuo kasvaa? (katsoo laskimensa näyttöä) Kutoseen. Ei. (naurahtaa) Sen näkkee tuolta alhaalta. (naurahtaa) Sitteku on nolla, niin se arvo on nolla.
When it approaches that minus one from the negative side, then it increases. Up to where? (looks at the screen of her calculator) Up to six. No. (short laugh) We can see it from there below. (short laugh) When it is zero, then the value is zero.

114 Veikko: Niin. (Anni naurahtaa) Mut siis miten me laitetaan se, että se ei saavuta ikinä sitä miinus ykköstä. Tai, tai siis pyssyy siinä ykkösessä siis.
Yes. (Anni laughs a bit) But how shall we write that it never reaches the minus one? Or, I mean, it remains at that one.

115 Anni: No laita
Well put it like that

116 Veikko: Ko seki taitaa olla, se taitaa olla jonkunlainen raja-arvo kans. Et se ei mee ykkösen alas. Voisko se olla?
Because it also may be, it may be some kind of limit as well. That it won't go below one. Could it be so?

117 Anni: Niin.
Yes.

118 Veikko: x-akselin suuntainen
In the direction of the x-axis.

119 Anni: (naurahtaa)Raja-arvo.
(short laugh) Limit.

120 Veikko: Niin.
Yes.

During turns 82 – 87, 91 – 104, 106 – 111 and 113 – 114 Jenni is working alone with her calculator. At other times she is following the discussion of Anni and Veikko.

There is no clear and authoritative leader in the group. Veikko and Anni are investigating the function in a collaborative way and trying to achieve a consensus. This is a typical feature of the participation structure in this group. During the episode, Jenni spends a great amount of time working alone with her calculator. Sometimes she is listening to the conversation between Veikko and Anni, and only once does she participate in the discussion. Her withdrawal in this episode is somewhat more extensive than usual. Normally she follows the discussions more closely and works along with Anni and Veikko when they are drawing graphs or writing down conclusions. When she rarely expresses herself, as in turn 90, she often speaks timidly with a low voice.

From utterances 79-84 it can be seen that Anni slightly and in a subtle way tries to direct the course of the working of the group. On the other hand Veikko constantly produces many initiatives and mathematical ideas; look for example, turns 86, 87, 89, 91, 98, 104, 116 and 118. These are typical roles of the students. Sometimes Veikko's ideas are divergent and even fuzzy. It is not uncommon that he communicates his thoughts vaguely. Turns 86 and 116 are examples of the divergent ideas and turn 87 of the vague expressions. Often Anni has the task of discreetly accepting or rejecting Veikko's numerous ideas. In this episode the most advanced mathematical idea comes from Veikko in turn 116. But more often it is Anni, who constructs the ideas which have the potential of significantly enhancing the development of the topic. There are some episodes in the data, where Anni and Veikko slightly compete about leadership and a few episodes where Veikko is leading. The atmosphere, however, always seems to be friendly.

Generally speaking, in the conversations of this group the students listen to each other. But for Veikko it is harder than for the girls. It is not uncommon that he ignores the comments of the girls or interrupts them. In this episode Veikko, twice, takes the floor from Anni (110 and 116) and prohibits her from expressing herself. Normally Anni gives up and follows his line of thought, which was the case here, too. When Jenni utters her only turn in this episode (90), Veikko ignores her. But Anni's "ym" after Veikko's turn is directed to Jenni.

In this group the students expressed agreement, when they thought that the other speaker was right. In this particular episode Anni and Veikko do it in turns 92, 104, 114, 117 and 120. All together in the data the students of this group made 210 utterances conveying agreement. But the frequency of expressions of disagreement was 70, only one third of the number of utterances of agreement. In this episode turns 81, 99 and 105 show some kind of disagreement. Sometimes the girls seemed to be in trouble when they had to reject Veikko's numerous ideas. It was not easy to do it in a polite way. Occasionally they used the strategy of disagreeing by silence.

It was not common in the data that the students in this group justified their claims. By justifying I mean giving any kind of reason for one's claim instead of just expressing a statement. There occurred only 22 justifications in the peer interaction of this group. One of them is in this episode. In turn 107 Anni gives a reason for her claim that when x increases without any boundaries the values of the function "remain" at 1. She calculates the values of the function at x = 500 and x = 1000 and gets a result 1.0 from her calculator. (The exact value is a little bit greater than 1, but the calculator rounds the result to 1.0).

After the first four small group sessions Veikko had to be absent from few lessons. When he returned, the first task for the group was to construct methods for finding the equations of a tangent and a normal to a curve at a particular x-value. It was a year ago when Veikko had studied the equations of lines, but the girls had attended the course during the previous period, just a few weeks ago. Jenni had the notes from the previous mathematics course with her, and she seemed to have knowledge about the important methods and formulae. The typical participation structure of this group changed when Jenni had her chance of participating in the working of the small-group.

Episode 2

The group has succeeded in constructing the equations of a tangent and a normal to the curve $y = \frac{1}{3}x^3 + x^2 + x$ at x = -1 (problem 5). They are beginning to write a summary about their investigation. Jenni orders Anni to write the summary on the transparency. After a short and friendly debate Anni gives up and starts organizing the writing.

19 Anni: Tarvittiinko me tuota [epäselvää] (osoittaa sormellaan vihkoonsa)?
Did we need this [indistinct] (points with her finger to the notebook)?

20 Veikko: Kirjotetaan siihen ekana aluksi, että. Ekaksi pittää sijoittaa tuohon (osoittaa sormellaan vihkoon). Älä kirjota vielä, mutta keskustellaan tämä. (Anni ja Jenni naurahtavat.) Ekaksi sijoitetaan tuo x, miinus ykkönen, tänne alkuperäiseen lausekkeeseen, että saahan se y:n arvo. Sitten pitää saaha se x.
Let's write here, first to begin with. First we need to substitute that (points with his finger to the notebook). Don't write yet, but let's discuss this. (Anni and Jenni give a short laugh) First we substitute that x, by minus one, here in the original expression, so that we get the y-value. Then we need to find the x.

21 Jenni: Eikö, se on siis se kulmakerroin (osoittaa kohden Annin vihkoa).
No, but that's the gradient (points twice towards Anni's notebook).

22 Veikko: Ni, eikö tä, niin, niin että sijoitetaan tuo, niin saahan se kulmakerroin.
Yes, but what, yes, that if we substitute by that, so we get the gradient.

23 Jenni: Ei.
No.

24 Veikko: Eikö siis, tähän sitä ei tarvitse. (Tytöt nauravat, Anni pitää käsillään päätään.) Tarviiko sitä sijoittaa tähän? Pittää.
No, I mean, we don't need to substitute this here. (The girls laugh, Anni holds her head between her hands) Do we need to substitute this? Yes we do.

25 Jenni: Pittää. Siitä saa sen y:n. Mutta tämä on se (osoittaa sormellaan Annin vihkoa).
Yes, to get the y. But this is it (points with her finger to Anni's notebook).

26 Veikko: Ja sitten pittää saaha x:n. Sitten pittää derivoia tuo alkuperänen lauseke, että saahan se tangentin
And then we need to get the x. Then we need to differentiate the original expression to get the gradient

27 Jenni: Niin.
Yes.

28 Veikko: kulmakerroin.
of the tangent.

29 Jenni: Just (naurahtaa). Ja sen jälkeen.
Exactly (laughs a bit). And after that.

30 Anni: Eli ensin (ottaa kynän esille) se lait. Kirjoitanko mie, että sijoitetaan x = -1 yhtälöön, tuo (osoittaa sormellaan vihkoonsa)?
So, first we put it (takes a pen). Shall I write that we substitute x by -1 in the equation, to here (points with her finger to her notebook)?

31 Jenni: Joo.
Yes.

32 Veikko: Vai pittääkö tästä tehä yleinen? Vai koskien just tätä tehtävää?
Or, do we need to write this in a general way? Or just about this exercise?

33 Jenni: Osataanko me tehä se yleinen?
Can we do it in a general way?

34 Veikko: No sillä tavalla, että jos ekaksi sijoitetaan x:ä tähän alkuperäiseen (osoittaa Annin vihkoa) yhtälöön,
Well, so that if we first substitute x in the original equation (points to Anni's notebook)

35 Anni: Eikö kirjoitetaanko, että
No but, could we write that

36 Veikko: saadaan
we get

37 Anni: tuon, tuon tangentin yhtälö (osoittaa kohtaa vihkossaan). Ja sitte kirjoitetaan, että kulmakerroin saahan siitäkö siihen sijoitetan se, se
the, the equation of the tangent (points to a place in her notebook). And then, let's write that we get the gradient by substituting in the, in

38 Veikko: Derivoitu.
Differentiated.

39 Anni: Derivoitu, niin.
The, differentiated.

40 Jenni: Joo.
Yes.

41 Anni: (ryhtyy kirjoittamaan) Mitä, tangentin yhtälö on (opettaja tulee luokkaan).
(starts writing) What, the equation of the tangent is (the teacher enters the classroom).

Here we can see a much more active role from Jenni. She participates in organizing the group activity (the debate before the episode and turn 33). She discusses with Veikko about the meaning of the different results they had obtained during their problem solving (20 – 29) and she supports Anni's suggestions (31 and 40). Even though there were episodes in this part of the data where Jenni was not this active, she was at least following the discussions of Anni and Veikko with more attention and participating in the discussion every now and then.

At the end of this episode (30 – 41) we can see a pattern, though not very clearly, that was more common in the latter part of the data. Anni and Jenni joined together to guide the course of the small-group activity.

4.1.2 Social norms produced or negotiated in group A

In this group the participating students, Anni and Veikko, normally collaborated and discussed until they arrived to a taken-as-shared interpretation of the phenomenon. During the later phases of the course, when Jenni was participating in the discussions, she followed the pattern, too. Through their interactions the students were producing the norm: *When working in a small group, the students should try to arrive at a consensual conclusion.*

One of the very first teaching experiments of Cobb et al. (Cobb, Yackel and Wood, 1989) took place in a second grade classroom. The authors report how the teacher started renegotiating social norms. She insisted that when working in groups, the children try to solve problems in a cooperative manner and reach a consensus. Tatsis and Koleza (2008) studied social and sociomathematical norms in collaborative problem solving of pairs of pre-service teachers with no prior experience in Greece. They report the existence of the collaboration norm: "The participants are expected to reach a mutual agreement on the solution process and its features, i.e. the concepts and procedures included" (p. 96). Tatsis and Koleza hypothesized that the fact that their students were adults who had known each other before the experiment, influenced the establishment of this norm. I see it as inevitable that in a class of very young pupils the collaboration norm for small-group working needs to be established for the first time. But the students in my research were second year high school students. If not in mathematics, at least in other school subjects, they had experienced working in small-groups and the expectations for students in those situations.

Cobb (1995, p. 26) sees the behavior according to the collaboration norm to be essential for the occurrence of learning opportunities while working in small-groups. Tatsis and Koleza (2008) observed it to be the most vital norm for the thematic development in collaborative problem-solving.

The tasks given to the students in my experimental class were different than problems used in realistic mathematics education. I did not intentionally plan contextual questions for which the students could construct many different solutions. The students in my two small-groups didn't have that strong need for explaining their own solutions and thinking, or for comparing and justifying them, as the students in the experiments of Cobb et al. This fact has influenced why I did not find exactly similar social norms being negotiated in my class as Cobb et al. have reported.

While co-operating, the students in this group, especially Anni and Veikko, expressed their own thinking. And all the students were listening to each other. Through these interactions the students were producing parallels to two of the basic social norms for group discussions in inquiry mathematics (Cobb, Yackel and Wood, 1989, Cobb and Yackel, 1998, Yackel, 2001), namely that students are expected to explain their solutions and try to understand the explanations of others. In my data I could draw inferences about the social norms: *When working in small-groups, students are expected to* ***express*** *their own thinking.* and *In small-group discussions, students are expected to listen to the others.*

On one hand, the students in group A were expressing their thinking and listening to the others, but on the other hand, the right to express oneself and the obligation to listen to the others were somewhat unevenly distributed. Veikko was very eager to express his thinking, and often he had problems in listening to the girls. He interrupted and ignored many of Anni's ideas, who then gave up, and he contributed to Jenni's retirement from the discussions. Through these interactions, the students were producing the norm: *In small-group discussions those, who are more assertive than others have the right to express their thinking, and others have the obligation to listen to them.* Cobb (1995) describes the social relationships of second grade pupils, and develops, from his analysis, the sociological constructs of mathematical and social authority. The latter means a power imbalance when one child regulates the way in which the pupils interact as they do and talk about mathematics. To some extent, Veikko was exercising social authority in the discussions of this group.

Showing agreement was common in this group, and all the students did their part in it. Through these interactions the students were producing the norm: *When working in small groups, the students are expected to express agreement.* On the contrary, expressing disagreement seemed to be more difficult for the students. Lampert et al. (1996) write about the discomfort of disagreeing in her fifth grade classroom. When not being able to find a consensus, the students in one small-group finally agreed to disagree so that they did not have to continue their process of negotiation. Many of Lampert's students felt threatened by the disagreements they had experienced in the class, and which they were expected to express. One of Tatsis and Koleza's (2008) social norms, namely that of avoiding threat, comes close to the lack of disagreeing. Through their interactions the students in group A were producing the norm: *In small-group discussions it is not appropriate to show disagreement too clearly.*

In this group the students did not justify their claims. How Lampert et al. (1996) describe traditional mathematics lessons in America comes close to what I often saw in this group. "In most school mathematics lessons, teachers and students make one assertion after another about what is true or false, right or wrong." And, she continues, "w(W)hat makes an assertion true or right is the authority of the teacher, the textbook or a smart student" (p.737). In this group, however, the students tried to discuss until they found a consensus. Through not justifying and through these interactions they were producing the norm: *The acceptance of a mathematical claim is on the basis of social agreement.* Although, as high level students, the members of this group can be supposed to have a certain competence in mathematical reasoning, this finding shows that, in their discussions, claims were not justified on a mathematical, but a social basis. In this respect, the situation is similar to the initial classroom culture of a second-grade class in Yackel and Cobb's (1996) experiment, from which they started negotiating norms more compatible with inquiry mathematics.

4.1.3 Participation structure in group B

Unfortunately, in four of the six sessions in this group, one person was missing. Thus the collection of students working together changed from session to session. It was, however, possible to distinguish and describe many features of the typical participation structure across sessions. This was so, because an important feature of the participation structure in this group can be characterized by a juxtaposition of the roles of Mika and Juha. Mika was always present, and Juha was absent from the first session, only. The roles of Reijo and Pekka were not so dominant. Very often, they were following either Mika or Juha, or both. I am going to use two episodes to illustrate the typical participation structures in this group.

Graph 2 The graph of the function $\dfrac{\sin x}{x}$

Episode 3

Mika, Pekka and Reijo in group B have been investigating what happens to the values of the function $f(x) = \frac{\sin x}{x}$ when x approaches zero from the right and the left (problem 1). The boys had constructed the required sequences. They were beginning to consider what happens to the values of the function. The graph of the function is presented above (graph 2).

134 Mika: Siihen. (lukee monisteesta) Mitä tapahtuu funktion f(x), sinx arvoille kun x lähestyy nollaa oikealta puolelta? Entä kun y lähestyy, x lähestyy nollaa vasemmalta?
There. (reads from the worksheet) What happens to the values of the function f(x) = sinx when x approaches zero from the right side? What if x approaches zero from the left?

135 Pekka: Mikä sin (näppäilee laskintaan)?
What sin (taps his calculator)?

It occurs to Mika that they had not yet succeeded in helping Reijo to draw the graph of the function with his calculator. The boys spend a while in discussing the problem. But they can not make the calculator work properly.

153 Pekka: (lukee monisteesta) Mitä tapahtuu funktion arvoille, kun x lähestyy nollaa oikealta?
(reads from the worksheet) What happens to the values of the function when x approaches zero from the right?

154 Mika: [epäselvää] zoomailla vielä sitä (katsoo Pekkan kuvaajaa). Niin se kasvaa ykköseen. (Reijo tutkii Mikan laskinta.)
[indistinct] zoom it a little bit (looks at Pekka's graph). Yes, it increases up to one. (Reijo is studying Mika's calculator)

155 Pekka: Enter (näppäilee laskintaan).
Enter (taps his calculator).

156 Pekka: Mennee kai ykköseen.
Goes to one, I guess.

157 Mika: [epäselvää]
[indistinct]

158 Pekka: O.
Oh.

159 Mika: No ei se ehkä tuosta sitä vielä, just hullun lähellä sitä no, ykköstä.
Well, perhaps you cannot yet, crazily near to it, well, one.

160 Reijo: Ai missä? (tutkii vielä Mikan laskinta)
Where? (is still studying Mika's calculator)

161 Mika: Jos se on, no, siinä ykösen ko, ninkö y yks. Ni, leikkaako se siinä vai ei?
If it is, well, there at one, I mean y is one. So, does it cut it there or not?

162 Pekka: Haluaksie että mie todistan?
Do you want me to prove it?

163 Mika: Ehän tuo, eikö tuo, voiko sin yks olla mitenkään? Mikä se on? Eikse oo niinku epämääränen, niinkö ei määritelty?
That can not, isn't that, can sin one be anyway? What is it? Isn't it in a way undefined, I mean not defined?

164 Reijo: Miks tää ei anna sillon mitään ku mä laitan x:ksi, laitan x:ksi nolla, niin y:stä ei tule mittään?
Why doesn't it give me anything when I make x, make x zero, there is no y?

165 Mika: (ottaa lasimensa Reijolta) Näytä. … Hyvä kysymys. x c nolla, yhe, nolla pilkku yheksän, kaheksan, nolla, kolome kaks. Ootappa. Sen voi pistää niinkö tänne, että f1 ja x on nolla. Siis. Niin, miehän sanoin. Undef.
(takes his calculator from Reijo) Show me. … Good question. x c zero, nine, zero point nine, eight, zero, three, two. Wait a minute. You can put it here, so that f1 and x is zero. So. Yes, I told you. Undef.

166 Reijo: Mitä undef?
What undef?

167 Mika: Sitä ei oo siinä. (näyttää laskintaan Pekkalle, joka katsoo)
It does not exist there. (shows his calculator to Pekka, who looks at it)

168 Pekka: Mitä ei oo?
What does not exist?

169 Mika: No se ei oo yks.
Well, it is not one.

170 Pekka: Ei ookkaan.
No it isn't.

171 Mika: Niin. Se ei osu siihen ikinä. Niin.
Yes. It never hits it. That is the case.

172 Pekka: [epäselvää]
[indistinct]

173 Reijo: Niin, lähestyykö se ykköstä?
So, is it approaching one?

174 Mika: Se lähestyy ykköstä, mutta ei ikinä tuu yköseen.
It approaches one, but never gets there.

175 Reijo: Niin, mut se lähestyy.
Yes, but it approaches.

176 Mika: Niin. … Niin, eli kirijotetaan siihen, et siitä tulee. (lukee monisteesta) x lähestyy nollaa oikealta. Niin.
Yes. … So, let's write there, that it becomes. (reads from the worksheet) x approaches zero from the right. Yes.

177 Pekka: Eli x, eiku funktio lähestyy ykköstä. Niinkö?
So x, no, the function approaches one. Right?

178 Mika: Niin varmaanki. (kaikki kirjoittavat) Funktion arvot lähesssstyy, miten, lähestyy ykköstä. Ykö, ykköstä, kirijotetaan.
I guess so. (all the students write) The values of the function approacccchhh, how, approach one. Let's write one.

179 Pekka: Yhtä.
Number one.

189 Mika: [epäselvää] lähestyvät yhtä. Entä ku x lähestyy nollaa vasemmalta?
[indistinct] approach one. What if x approaches zero from the left?

190 Reijo: Sama juttu. Eikö?
It's the same. Isn't it?

191 Pekka: Aa.
Mm.

192 Mika: Ym. (kirjoittavat)
Mm. (the students write)

Sometimes this group found its direction in a democratic way. On those occasions many students contributed to the discussions and thus participated in leading the course of their working. But very often it was Mika, who guided the group towards mathematical discussions and establishing the task, normally in resonance with the others. In this episode, we can see Mika performing this kind of role. He starts the investigation about what happens to the values of the function (134), he acts an important part in the collective construction of knowledge (153 – 176) and he leads the group to writing down their conclusions (176 – 192). Often Mika helped Reijo and Pekka in their problems with following the working of the group. In this episode, Mika tries to figure out why Reijo could not draw the graph of the function with his calculator, and he answers Reijo's and Pekka's questions (164 – 192).

Normally Pekka and Reijo were trying to follow and co-operate with Mika, as in this episode. Seemingly they wanted to participate in the group work. They asked questions (160, 164, 166, 168, 173, 177 and 190 in this episode) and, for example, when it was their turn to write down the summary of the group on a transparency, they were actively organizing the group work and requiring that Mika should tell them what to write down. The role taking of Pekka was, perhaps, the most inconsistent in this group. There were times when he fiercely explained his thinking to others and when he, supported by Mika, led the group to realize an idea he had discovered. But, sometimes, he just tried to follow the others or did not even bother to do that. Occasionally he could chat with Juha about topics other than mathematics. Reijo was more conscientious and more faithful in following the leader. In this group, he had the greatest frequency in asking questions when he did not understand something. He might get an answer, but sometimes the others were so preoccupied with their own thoughts and discussions that they didn't listen to him even if Reijo insisted with his questions. A few times Reijo tried to calm down a beginning quarrel or asked the others to keep focused on the task.

Another strong role in the group, in addition to that of Mika, was played by Juha. The following episode illustrates typical features of the participation structure in the group when he was present.

Episode 4

The boys in group B had been given a task of finding the equations of the tangent and the normal to a third order polynomial function f(x) = $\frac{1}{3}x^3 + x^2 + 1$ at x = -1 (problem 5). They had differentiated the function and obtained $x^2 + 2x$. Mika is starting the discussion about how they should continue. The boys begin by substituting x by -1.

232 Mika: No mitäs sitte? Siinon nyt paperilla.
Well, what next? There we have it on the paper.

233 Pekka: Sitten pistetään x:n tilalle yk, miinus ykkönen.
Then we put, instead of x we put minus one.

234 Juha: Mitä, mitäää, mitääs myö sitte tehhään?
What, whaaat, whaaat are we going to do then?

235 Pekka: No sitte me pistetään sen x:n tilalle yksi.
Well, then we put one instead of x.

236 Mika: No ko ei se niin. Onkohan se niin heleppo?
But it's not that. Can it be that easy?

237 Reijo: Ykönen on kak, miinus yks potenssiin kaks on yks.
Number one is two, minus one to the second power is one.

238 Juha: Ykkönen? Miinus ykkönen.
Number one? Minus one.

239 Pekka: Niin, niin, niin (äänensävy vähän ärtynyt).
Yeah, yeah, yeah (with an irritated tone).

240 Juha: Pekka, sillä on merkitystä.
Pekka, it does matter.

241 Pekka: Ei.
No, it doesn't.

242 Mika: Yks plus miinus kaks elikkä miinus yks.
One plus minus two that makes minus one.

243 Pekka: Miinus yks toiseen plus kaks kertaa minus yks.
Minus one to the second plus two times minus one.

244 Reijo: Nii-i.
So it is.

245 Mika: Miinus yks.
Minus one.

246 Juha: Tulee ihan ala-aste siitäki, että yks plus kaks plus
It's like in primary school, this one plus two plus

247 Pekka: Yks plus kaks
One plus two

248 Reijo: No niin.
Okay, boys.

249 Pekka: kertaa miinus yksi on niinkö kolome (Mika kumartuu Pekkan puoleen).
times minus one is about three (Mika bends towards Pekka).

250 Reijo: Laskekaa.
Do your job!

251 Juha: (osoittaa sormellaan Mikaa ja Pekkaa) Nuillekko sie puhut?
(points with his finger at Mika and Pekka) Are you talking to them?

252 Mika: Se on miinus yks.
It's minus one.

253 Pekka: Se on miinus kolome. (selittä Mikalle) Tuosta tulee yks. Yks plus kak, eikö sitte tuosta tulee, öö miinus kaks. Ja sitte ne ku plussaa yhteen, niin siitä tulee miinus yks (Reijo naurahtaa).
It's minus three. (explains to Mika) This makes one. One plus two, no, that makes then, oh minus two. And if you sum them, that makes minus one (Reijo gives a short laugh).

254 Mika: Niin.
That's it.

255 Pekka: Niin, miehän sanoin (kaikki naurahtavat).
Told you (everybody laughs).

256 Juha: Pekka, menään ala-asteelle. Etkö ossaa laskea, että -1 + 2?
Pekka, let's go to the primary, shall we? Can't you work out -1 + 2?

257 Reijo: Mitäh?
What?

258 Juha: Toisinpäin, toisinpäin, joo.
The other way, the other way, yeah.

259 Pekka: -1 + 2, mitä se on?
-1 + 2, what's that?

260 Juha: [epäselvää]
[indistinct]

Mika leads the group into a discussion about the meaning of their result.

261 Mika: No ni. Mitä tuosta oli hyötyä nyt? Ei mittään. Mikä tuo on?
So. What use do we have for that? Nothing. What's that?

262 Reijo: No äläpä nyt vaivu epätoivoon.
Well, don't despair.

263 Juha: Mikä tuo on? No
What's that? Well

264 Pekka: Se on se kulmakerroin.
It's the gradient.

265 Juha: Oisko kulmakerroin?
Could it be something like gradient?

266 Mika: No ei.
No way.

267 Pekka: Kulmakerroin on kaks. (Juha ottaa laskimensa esille.)
The gradient is two. (Juha takes his calculator)

268 Mika: Kulmakerroin on yks. Musta tuntuu, että tuo on se kohta, missä se on tangentti, ja tuo on se [epäselvää], miinus ykösesä.
The gradient is one. I think that this is the x at which the tangent is and this is the [indistinct], at minus one.

269 Juha: (piirtää tangenttia kohtaan x = -1) Math, tangent, tangent, tangent at miinus ykkösessä.
(is drawing a tangent at x = -1 with his calculator) Math, tangent, tangent, tangent at minus one.

270 Pekka: Se yhtälö on kato silleen, että
The equation is, you see, like that

271 Juha: (näyttää laskintaan Mikalle, viheltää) Miinus yks x.
(showing his calculator to Mika and whistling) Minus one x

272 Mika: Ym.
Um.

273 Juha: Niin. Oisko se se kulmakerroin?
Yes. Could it be the gradient?

274 Mika: Miksi se ois?
Why would it be?

275 Juha: Hä?
What?

276 Mika: Miksi se ois? Voishan se niin ollakki. Kyllä se on se. (Juhalla voitonriemuinen ilme)
Why would it be? Well, I guess it could be. Yes, it is. (Juha smiles triumphantly)

277 Reijo: Hei mitä tässä tehtävät sillon sano? Ne pitää laskia.
Look, what are the exercises she said? We have to do them.

278 Pekka: Tuosson vaan se yhtälö.
That's only the equation.

279 Juha:	[epäselvää] tyhmät ratkaisseet. *[indistinct] idiot have solved.*
280 Mika:	Siinon kulmakerroin muuten (Reijo ottaa tehtäväkirjan esille). *Here we have the gradient by the way (Reijo finds his book of exercises).*
281 Pekka:	Eise, eise voi pittää paikkaansa. *No, no it can't be true.*
282 Mika:	No ohase. Tuohan on kulmakerroin. *But it is. That is the gradient.*
283 Pekka:	Mikä sitte tämä on tässä (näyttää sormellaan kohtaa Mikan vihkosta). *What is this here, then (points with his finger to a place in Mika's notebook)?*
284 Juha:	Perhana täälon vihkosa muistiinpanot miten kulumakerroin lasketaan. Kattokaa sieltä (selaa omia muistiinpanojaan). *Damn, here in the notebook it says how to find the gradient. Have a look there (leafs through his own notes).*
285 Mika:	Sehän on [epäselvää], kulumakerroin, kulumakertoimen yhtälö. *Well, it is [indistinct], gradient, the equation of the gradient.*
286 Pekka:	Yhtälö. Yhtälö. Niin. No niin. Kyllähän mie sen tiiän. *Equation, equation. Yeah. So, well. I do know that.*
287 Mika:	Niin just. [epäselvää] *Exactly. [indistinct]*
288 Reijo:	Niin mikä? *What is?*

Juha acted in a more individualistic way than Mika. He spoke in ways and at times that did not always seem to fit the discussion of the group. In this episode, see turns 234, 25, 269 and 284. Usually he participated actively in the mathematical discussion. But he also very often gave remarks and comments that were not relevant to the mathematical theme. He led the interest of others towards topics like computers, programming and games as well as free time activities. In this episode he is remembering his studies in the elementary school (246 and 256). Often he bullied the other members of the group and praised himself trying to support his own status, in this episode see turns 256, 276 and 279.

There is certain competition in the data of this group, especially between Mika and Juha. It is seen in this episode, too. Mika asks what use they have for the value of the derivative function at x = - 1 (261). Pekka and Juha state that it is the gradient (264 – 265). Mika first rejects the suggestion (266), perhaps because of the arrogant style of Juha's comment (265). But when Juha draws the tangent with his calculator and shows the equation to him, Mika admits that the boys (or Juha) have it right. At this point Juha is smiling triumphantly (276). Normally, during the small-group work, both Juha and Mika expressed intelligent mathematical ideas. However, Mika was accepted as a mathematical authority by Pekka and Reijo, but Juha was not.

The competition between Juha and Mika occasionally turned the mathematical discussion into a social game of disputes and boasting. In those situations the mathematical discussion was not honest. It was a means of trying to beat the opponent and show one's superiority. Generally, it seems to me that Mika was less prone to this kind of social game than Juha. If he had good reasons, he could change his mind in favor of Juha's idea, as in episode 4, in turns 271 – 276.

Very often, as we can see in episodes 3 and 4, the students in this group were working together to reach a common conclusion about the topic at hand. But there were more deviations from that in this group than in group A where, usually, only Jenni's retirement was an exception to the rule. One striking reason for this was especially Juha's, but also Pekka's, individualistic behavior when they did not adjust their actions to the course of the group work. Many times, therefore, the discussion in the group was split into two between pairs of students. In addition, Juha often bullied the others, who then turned to work without him. It seems to me, however, that the behavior of Juha was not the only reason to disturb collaboration in this group. I looked specifically at the first session, when he was absent. There seemed to be certain kind of harmony in the working of the group. But even there the collaboration broke down a couple of times due to the individualistic behavior of Pekka or Mika. The session ended so that Pekka copied a sentence from the textbook and wrote it in his notebook. It suited very well to be a summary of the group work, and when I read it, I thought it to be the conclusion of the whole group. The other boys were confused when I asked the group to write down their conclusion on a transparency. They asked: What conclusion? Also the competition between the students and their difficulties in listening to others contributed to the lack of collaboration.

Often the boys in this group were listening to each other. But in many occasions they had problems in this respect. In the four latter sessions the boys frequently interrupted each other. Sometimes there was a real competition for the floor, everybody trying to contribute to the discussion and not listening to the others. It seems to me that Juha triggered the phenomenon. During the first session he was absent. And during the second session he did not quite follow the other boys who already had some experience of limits. The frequencies of interrupting others were low in those sessions. (1. session: Mika 3, Pekka 1 and Reijo 6, 2. session: Juha 3, Mika 2, Reijo 6). It may also be that the boys were working for the first times in front of the camera and tried to behave "properly". During the four last sessions it was Juha, who had the greatest frequency (85) in interrupting the other students. But the others were also able to respond to the challenge (Mika 57, Pekka 36 (one session absent), Reijo 42 (one session absent)). In this group, interrupting others was more symmetrical than in group A, where it was Veikko who superseded the girls in this habit (Veikko 30, Anni 14 and Jenni 1). Two of the boys had more problems than the others in being listened to. Maybe because of his disruptive behavior, the other boys sometimes ignored Juha's good ideas and did not take him seriously. Reijo asked a lot of questions to be able to follow the working of the group. His questions sometimes remained unanswered.

In this group, the students used more versatile strategies for mathematical discussion than in group A. They showed both agreement and disagreement, the latter almost as frequent as the former (148 utterances in agreement and 132 expressing disagreement). They justified their mathematical claims much more often than the students in the other group (76 justifications in group B and 22 in group A). Again, I have to remark that justifying in my analyses meant giving some kind of reason for one's claims. This is not to say that there were no problems in this respect in group B. Especially in one session, the introduction to the derivative, only 3 justifications were given and none of them was based on properties of mathematical objects. During that lesson the decisions were made by authority of one or two students.

4.1.4 Social norms produced or negotiated in group B

When collaborating, the students in this group were producing the norm: *When working in a small group, the students should try to arrive at a consensual conclusion.* Mika's democratic leadership and his caring for Reijo and Pekka also contributed to this. However, through their individualistic behavior, some of the boys, especially Juha and Pekka, were often breaching the norm. Tatsis and Koleza (2008) describe the phenomenon among their pairs of pre-service teachers. According to them, the absence of the norm, even in small parts of the discussions, led to tension and disorder. It was as if there were two independent persons in the setting instead of a pair working in a collaborative way. The students in my experimental course were not really accustomed to collaborating during mathematics lessons. Another reason for the frequent breaching of the norm in this group might be the style of friendship interaction typical of boys (Maltz and Borker, 1982, Partanen, 2007).

While cooperating, the students were producing the parallels of the social norms of explaining one's solutions and trying to understand the solutions of others (Cobb, Yackel and Wood, 1989, Cobb and Yackel, 1998, Yackel, 2001): *When working in a small-group, the role of the students is to* ***express*** *their thinking.* and *When working in small-groups, students are expected to listen to the others.* But, by interrupting and not listening to the others, the students also produced the norm: *In small-group discussions assertive students have the right to express their thinking even at the expense of not listening to the others.*

Through the way Mika helped Pekka and Reijo who were seeking it, the students in group B were producing the norm: *In small-group activity the role of the student with mathematical competency is to listen to those who have got problems and try to help them.* Salo (1999) describes how in a Finnish first grade classroom in Rovaniemi the teacher used some girls in the classroom as assistant teachers. It seems to me that in this group Mika was, to a certain extent, acting the role of the assistant teacher. Mulryan found in her unpublished dissertation (Good et al., 1992) that in the interviews after working in small-groups both the students and teacher mentioned help-giving and/or help-seeking as expected behavior in small collaborative groups. Low-achievers asked the most questions and high-achievers showed more information-giving behavior. Kumpulainen (2002) sees tutoring as a desired social process during communal problem solving.

As in group A, the students in this group were producing the norm: *In small-group discussions students are expected to show agreement*. In addition, the students in this group were producing, to a greater extent than in the other group, the parallels to vital norms for inquiry mathematics (Yackel, 2001, Yackel and Rasmussen, 2002): *In small- group discussions students should justify their claims.* and *In small-group discussions students are expected to express disagreement.* The justification norm was also found in Tatsis and Koleza's research on pre-service teachers' problem solving (2008) although the students did not have much prior experience about problem solving or working in small groups in mathematics. Hunter (2007) suggests that arguing and disagreement are important foundations for further shifts toward mathematical argumentation. On the other hand, there were times when the boys in this group were producing the norm: *The justification of mathematical claims is in the authority of the speaker.* Being good at mathematics seemed to be a way of posturing in this group. Through their competition and disputes, the students were producing the norm: *A student should show competence and confidence in mathematics.* The social status of a student in this group was tied to the competence he was showing in mathematics.

4.1.5 Participation structures in the teacher-students interactions

I visited both groups A and B every now and then and tried to follow their progress when the students were working with the assignments. Those situations clearly caused a different phase in the interactions of the small-groups. In this section I am going to describe typical and interesting features of the participation structures during those periods.

Episode 5

Group A was asked to find the equations of a tangent and a normal to a third order polynomial function at x = -1. Veikko suggested that they should differentiate the function. But, he continued, the function thus obtained would not be a tangent. Anni expressed her idea that they should first differentiate the function and then substitute x by -1. The group acted according to her idea even though Veikko kept on insisting that they should differentiate the function for a second time (to obtain a line). At this point, I joined the group.

60 Opettaja. Yhm. Ootteko päässy ajatuksessa?
Teacher: Ur-m. Have you any idea?

61 Veikko: Ei olla oikeastaan. Ko me vaan derivoitiin se muttako ehe (naurahtaa).
Not really. Coz we just differentiated it but then (gives a short laugh).

62 Opettaja: Joo. Eli tuota tämä on nyt se derivaattafunktio (näyttää sormellaan Veikkon vihkoa) eli se kulmakerroin funktio.
Okay. So, this is the derivative function (points with her finger to Veikko's notebook), or the gradient function.

63 Veikko: Niin.
Yes.

64 Opettaja: Ja mitäs hyötyä tuosta nyt on?
And what use do we have for it?

65 Veikko: Niin.
Well.

66 Anni: Voiko sen sitte (naurahtaa) laskea siitä sen
Can you then (gives a short laugh) find the

67 Veikko: Niin, että jos me sijotettas tuo miinus ykkönen tähän ja sit saatas se kulmakerroin.
Yeah, if we substituted this here by minus one and we would get the gradient.

68 Opettaja: Saisitte tangentin kulmakertoimen sitä kautta, juu. (Opiskelijat ryhtyvät kirjoittamaan vihkoihinsa.)
You would get the gradient that way, yes. (the students start writing in their notebooks)

69 Veikko: Okei. Mitä me sitte tehään?
Okay. What shall we do next?

70 Opettaja: Pitäs saada yhtälö sille tangentille. Tangenttihan on suora.
You need an equation for the tangent. Well, a tangent is a line.

71 Veikko: Aivan. (Opettaja poistuu.)
Exactly. (the teacher exits)

What was striking in the interactions between me and this group was that normally it was Veikko who represented the group and communicated with me. He expressed his own and others' thinking and asked me questions. In this episode, Veikko interrupts Anni in her sole attempt to participate in the discussion (66 – 67), and explains to me an idea which was originally developed by Anni. Sometimes Anni was participating in the discussions almost as much as Veikko, but the majority of the episodes are like this one. Very seldom did Jenni speak when I was present. It has to be noted that I did not do anything to try to include the girls in the discussions. I communicated with the persons who volunteered to interact with me. In group B, Mika most often discussed with me. But the pattern was not as clear as in this group.

In the discussions between me and the two small groups, we normally were listening to others. But the assertiveness of Veikko, as well as Mika and Juha in group B was also seen in the observation that these students often interrupted other speakers, even me. Another reason for not listening was the individualistic behavior of one or two students. Sometimes, when I was concentrating on trying to understand what a student had just said or written, at the same time I ignored what another student was trying to say.

In our discussions, the participating persons expressed agreement with each other. But, on the contrary, I was the only person to disagree (11 times in the group of girls and 9 times in the group of boys). One exception to the rule was Mika, who challenged my ideas three times.

In the teacher-students interactions we justified our claims altogether 17 times (by some kind of reason), 8 times in group A and 9 times in group B. I justified 13 times and four students did so once. It is obvious that neither I nor the students were accustomed to giving reasons in our face-to-face discussions.

My data includes a few moments when I acknowledged the need to act in a new way and justify my claims. These situations happened when I disagreed with a solution that the students had developed. I understood that, when investigating mathematics, it is not appropriate for a teacher to use her authority and just tell the students their solution is not right. The following episode presents such a situation.

Episode 6

Group B was working with a time-distance graph (problem 4). They had been asked to evaluate the instantaneous velocity at one second. They had drawn a chord from x = 0 to x = 1 and calculated the area between the chord and the x-axis in that interval. They had interpreted this area to be the instantaneous velocity. I could not have imagined a solution like this. I was astonished, and the students could see it.

319 Opettaja: Joo-o. (Anti ääntelee o-ouu, Reijo naurahtaa)
Teacher: Yee-es. (Juha says uh-o, Reijo gives a short laugh)

320 Pekka: Se on sanottu fysiikan kirjassa semmonen.
It is written in the physics book.

321 Opettaja: Eli nyt, nyt tuota, joo. Nyt mennee sillailla pikkusen niinku epäloogiseksi, että jos se on kulmakerroin se he, keskinopeus, niin, niin tuota nyt kulmakertoimella ja pinta-alalla on todennäköisesti eri yksikkö. Eli tuota, tuota, tuota. Mitenkähän me tästä voitas
Well, now, yes. It is a bit non-logical, that if the gradient is the inst, average velocity, then, well, gradient and area most probably have a different unit. So well, well, well. How could we

322 Mika: Eihän me kulmakerrointa käytetty siinä.
But we didn’t use any gradient there.

323 Pekka: Ym.
Um.

324 Juha: Siksi minua vähän epäilyttääki.
That's why it looks suspicious to me.

325 Opettaja: Me pohdittiin sitä, että ne sekantin kulmakerroin tarkotti sitä keskinopeutta. Ootteko te laskenu näitä sekantteja? Tai piirtäny näitä sekantteja?
We were considering that the gradient of the chord means average velocity. Have you evaluated these chords? I mean drawn these chords?

326 Peteri: Ollaan.
Yes we have.

327 Mika: Ollaan.
We have.

328 Opettaja: Millä tavalla te, minkälaisia sekantteja piiirsitte?
How did you, what kind of chords did you draw?

329 Pekka: No, Mika on
Well, Mika has

330 Mika: No mullon ne.
I've got them.

In this new and surprising situation I quickly developed what actually was a good justification; gradient and area have different units in the time-distance axis, so both of them cannot be velocities. But the way I expressed my justification was clumsy. After hastily expressing it, I went on and started guiding the working of the group to the "right track". I was neither accustomed to justifying my claims nor very skilled in it. The situation would have offered a fine opportunity to teach the students that the representation of a physical quantity in a graph must be compatible with the unit of the quantity. And, this moment could have been a critical moment for negotiating the need for justifying one's claims.

4.1.6 Social norms produced or negotiated in the teacher-students interactions

I was accepting the role of Veikko as the representative of the group, and I was not trying to invite the girls into our discussions. Through these actions, in addition to all the students in group A, I participated in the production of the norm: *The students who are assertive have the right to express their thinking and the others have the obligation to listen.* By interrupting others and not listening to them, the three boys, Veikko, Juha and Mika, were also producing this norm.

By listening to others and showing agreement we were producing parallels of important norms in inquiry mathematics (Cobb, Yackel and Wood, 1989, Cobb and Yackel, 1998, Yackel, 2001): *In classroom discussions participants are expected to listen to others.* and *In classroom discussions participants are expected to show agreement.*

The asymmetry in disagreeing, in that I was (with three exceptions) the only person to do so, and the lack of justifying suggest that we were acting according to, and producing the norm: *The justification of mathematical claims is in the authority of the teacher.* Traditionally in my lessons, I first derive the important ideas to be studied in teaching from the front, and show some examples, after which the students practice with exercises. When I introduce new topics, I try to give my students plenty of formal and informal justifications. But in my more intimate discussions with them, I just advise them with instructions and principles for solving tasks, and my approval justifies the solutions. I now see from Lampert et al. (1996) that this way of acting leads to patterns of interaction, and corresponding beliefs, that the authority of the teacher makes an assertion true.

4.1.7 Summarizing and comparing the participation structures

In the peer interaction of both of the groups, the students were trying to act according to the collaboration norm. Although they were not accustomed to studying in small-groups during mathematics lessons, they had experienced collaborative working methods in other school subjects. In studies with small children (Cobb, Yackel and Wood, 1989, Cobb and Yackel, 1998) it is natural that social norms for small-group work need to be established anew. But in my small-groups the students were almost adults (17 years old) and they had prior experience of different situations when studying at school. The collaboration norm was already established to a certain extent. However, especially in group B, there were problems concerning the norm. In this group the students acted more from an individual basis than in group A. During the experimental course I noticed problems in this respect in some other groups as well.

I was surprised to see how, in both small-groups, different students had different rights to express their thinking. Including me, we all contributed to that. Veikko in group A prohibited Anni from expressing her ideas and he was the main reason for silencing Jenni. In those situations the girls did not protest; they retired. In my presence, both I and the girls let Veikko act as the representative of the group. In group B Mika and Juha acted in the most dominant roles and frequently interrupted others. Reijo and Juha sometimes had difficulties to be listened to, though for different reasons. The use of small-groups in instruction offers possibilities for students to be active in learning (Good et al., 1989 – 1990, Good et al., 1992, Kumpulainen, 2002). But very often peer discussions are not democratic in that some group members exert authority and dominate others. It is one of the major challenges in developing the use of peer collaboration in instruction, that participation in the small-groups is made more equitable (Good et al., 1992, Cohen, 1986, Richmond and Striley, 1996.) Status, prior achievement in mathematics, and gender have been reported to be connected with dominant behaviour in peer-groups (Good et al., 1992).

Lack of justifying one's mathematical claims seems to be a pattern in my data, both in the peer interaction of the two small groups and our teacher-students interaction. In our discussions, mathematical claims were accepted on the basis of authority or social agreement. Jiménez-Aleixandre et al. (2000) report how, in their small-groups of high school students discussing science topics, agreements among students were often reached by one or more group members exerting authority, or on the basis of the "majority rule". Many research reports from experimental classrooms where the inquiry mathematics culture is being established describe this kind of situation as the starting point (Yackel and Cobb, 1996, Lampert et al., 1996). It seems to be an undesirable result of patterns of interaction in traditional mathematics classrooms (Bauersfled, Krummheuer and Voigt, 1988). However, if we are going to build creative cultures in classrooms, the situation must be changed. During the experimental course, I had a certain anticipation of the need for change, but at that time I did not understand the role of norms in the classroom, and I did not have time and energy enough for reflection to become consciously aware of that.

Table 1 The frequencies of justification, utterances of agreement and disagreement in the two small groups

group	justification	agreement	disagreement
A: Anni, Jenni and Veikko	22	210	70
B: Juha, Mika, Pekka and Reijo	76	148	132

In group B the students justified their claims more often than in group A (table 1). They also challenged each other by disagreeing much more often, while in the other group disagreeing too clearly was not appropriate. On the other hand, in group A the students showed agreement with each other more often than in group B. When comparing the groups, one could characterize that in group A there was an atmosphere of agreeing and in group B an atmosphere of arguing. The boys in group B even had disputes with each other. Nelson and Abound (1985) report that in their study friends explained their opinions and criticized their partners more often than non-friends. I know that the students in group B were mostly friends in their free time, but I have not the information available about the students in group A. On the other hand, in their review of the use of small-group discussions in science teaching Bennet et al. (2010) report on three studies (De Vries et al., 2002, Keys, 1997, Tolmie and Howe, 1993) which identified clear differences in interactional styles with all-male groups and all-female groups. Students in the all-male small-groups tended to confront differences in their individual perspectives whilst all-female groups searched common features of their predictions and explanations in order to avoid conflict. The differences in the atmospheres of the groups A and B may reflect the different styles of interaction of girls and boys. To a certain extent, group A can be seen to represent an all-female group. This is so because Veikko, the male member of the group, used strategies of talk typical of both boys and girls (Partanen, 2005, Partanen, 2007).

4.2 Participation structures and social norms in the two small groups in terms of knowing and learning

4.2.1 Participation structures in terms of knowing and learning: peer interactions

When investigating mathematics in small-groups, the students were given questions to be discussed and problems to be solved so that they would together, if possible, construct important mathematical ideas, or parts of them. Thus, through the assignments, I conveyed an implicit expectation that the students should express their own authentic thinking and create new mathematics. Often my students fulfilled my expectations. However, they were not accustomed to studying mathematics through an investigative approach. Every now and then, especially in the beginning of the experimental course, but also later, the students acted according to traditional norms which were not relevant in the new situation.

Episode 7

Group A was asked to find out what happens to the values of the function $f(x) = \frac{\sin x}{x}$ when x approaches zero from the right and from the left (problem 1). So far the group had concluded that the function has not got a value at $x = 0$ and that there is not a point on the graph at $x = 0$. The students had constructed the required sequences and in the beginning of the episode they were discussing what happens to the values of the function when x approaches zero from the right and from the left. The graph of the function is presented earlier (graph 2).

152 Veikko: Pistetäänkö me vaan, et se on nouseva? Ja sitte ku se on
Shall we just write down, that it's increasing? And when it is

153 Anni: Mut sehän on sitte kummastaki.
But it is, then, from both directions.

154 Veikko: Ku se lähestyy nollaa oikealta, niin se tulee täältä suunnalta (liikuttaa kynäänsä pitkin Annin vihkossa olevan koordinaatiston x-akselia oikealta vasemmalle).
When it approaches zero from the right, it comes from this direction (moves his pen along the x-axis in Anni's notebook from right to left).

155 Anni: Ym.
Um.

156 Veikko: Eikö me yleensä aina ajatella, että se niinku laskee ku me katotaan sitä oikealta vasem, vasemmalta oikealle? Mulla on aivan sekasin kaikki.
Don't we normally think, that in a way it gets lower when we look at it from right to left, from left to right? I am totally confused.

157 Anni: (naurahtaa) Ni. Eli sillonhan se niinku laskee.
(gives a short laugh) Yeah. So, then it, like, decreases.

158 Veikko: Mutta justiin siinä sanottiin sillonku x lähestyy nollaa oikealta.
But, it says that when x approaches zero from the right.

159 Anni: Väsemmältä (naurahtaa).
From the left (gives a short laugh).

160 Anni: Mutta jos se, sillonku se lähetyy vasemmalta, sillon se on ainakin nouseva. Eikookki?
But if it, when it approaches from the left, then at least it is increasing. Isn't it?

161 Veikko: Joo.
Yes.

162 Anni: Miten jos se oikealta? Jos se on ihan sanatarkkaa, niin se on oikealta
What if it comes from the right? If we take it literally, then from the right it is

163 Veikko: [epäselvää] Kasvava.
[indistinct] Increasing.

164 Anni: Kasvava.
Increasing.

165 Veikko: Onko se aidosti kasvava?
Does it increase in a proper way? (a term in Finnish school mathematics)

166 Anni: (naurahtaa) On se varmaanki.
(gives a short laugh) I guess so.

167 Anni: Mut jos se oikealta
But if it comes from the right

168 Veikko: Mutta siinä on jo mun mielestä. Mie en oikein tajua tuota käsitettä. Sillonku se lähestyy, eikö sen pitäs mennä niinku poispäin siitä (piirtää kynällään ilmaan kuvaajan muodon).
But I think. I don't understand that concept. When it approaches, shouldn't it in a way go away from it (draws with his pen the shape of the graph)?

169 Anni: (ottaa laskimensa esille) Niin, mutta kato minkälainen ruutu tää on (muotoilee kynällään käyrää vasemmalta oikealla) [epäselvää]. Saanko mie kattoa mitä sullon tuola (ottaa Veikkon laskimen ja tekee sille jotakin).
(takes her calculator) Yes, but look at my picture. (follows the graph with her pen from left to right) [indistinct]. Can I see what you have there (takes Veikko's calculator and does something with it).

170 Veikko: (Jennille) Funktio on kasvava vai mikä on kasvava? (naurahtaa)
(to Jenni) The function is increasing or what is increasing? (gives a short laugh)

171 Veikko: Ko se on periaatteessa kasvava, mut se periaatteessa pitäs vähentyä.
Coz, in principle it is increasing, but in principle it should decrease.

172 Anni: Niin. Se vähän riippuu, kummasta suunnasta sitä katotaan.
Yes. It depends from which direction you look at it.

173 Veikko: Hei, pistetäänkö siihen vähenee? Koska jos me ajatellaan, et me tullaan täältä suunnalta, niin se vähenee.
Listen, shall we write that it increases? Because if we think that we are coming from this direction, it increases.

174 Anni: Niin.
Yes.

In the beginning, Veikko and Anni arrive to a conclusion that the values of the function increase when x approaches zero from the right or from the left (152 – 155). This is a correct statement, and could be elaborated further. But Veikko finds their conclusion troublesome (156). Before this lesson the class had been studying increasing and decreasing functions. I had emphasized that in those tasks, we always let the x-value increase, which means we move from the left to the right. The students discuss that, if they think of the question on the paper, they would answer that the function increases in both cases. Veikko finds it more tempting to follow my instructions from the previous lesson (156 and 173) and with Anni's approval they arrive to a conclusion against their own reasoning. A similar phenomenon happened three times in this group.

In the other group, Mika, Pekka and Reijo were beginning the same investigation.

Episode 8

The students had drawn the graph of the function $\frac{\sin x}{x}$ (graph 2) with their calculators. Reijo had not yet succeeded in getting the graph visible. Mika finds his formula booklet and starts leafing through it. He is a little bit unsure whether they are allowed to use it in the new situation.

36 Mika: Kai se on mahollista ottaa ja kattoa niitä juttuja (ryhtyy etsimään taulukkokirjaa).
I guess we're allowed to take and read from there (starts searching his formula booklet).

37 Reijo: Ai mitä?
What?

38 Mika: Tai silleen ki, [epäselvää] kirijasta (selaa taulukkokirjaansa).
Or from the bo, from the book (leafs through his formula booklet).

39 Mika: Täälon jotaki raja-arvosta.
Here it says something about limits.

After a while Mika tells Reijo to search for information from the textbook.

64 Mika: Niin. Mitä me voiaan tuosta niinkö nähä? Mikä on funktion sinx raja-arvo. Aukaseppa Reijo sie kirija. (naurahtaa)
So. What can we sort of see in that? What is the limit of the function sinx? You, Reijo, open the book.

65 Reijo: Ai mikä?
What?

66 Mika: Mikä on raja-arvo, sielä joku, mikä se on. (selaa taulukkokirjaa, Reijo ottaa oppikirjan esille)
What is the limit, that, there? What it is? (leafs through the formula booklet, Reijo opens the textbook)

67 Pekka: Mikä on raja-arvo.
What is a limit?

68 Mika: Onko tämä niinkö derivoida juttuja? Vai mikä tämä on? (Reijo naurahtaa, Pekkakin hakee kirjansa)
Is this differential stuff? Or where are we? (Reijo gives a short laugh, also Pekka opens his textbook)

69 Mika: Hei miks täälä ei oo mitään raja-arvoa?
Look, why is there nothing about limits here?

70 Reijo: Tästä ei tajua mittään.
You can't understand anything about this.

71 Mika:		Jossaki täälä kyllä oli semmonen. *There was something here.*
72 Pekka:		Noo, sivulla viis kaheksan. *Well, on page fifty eight.*
73 Mika:		Mikä siinä lukkee? Raja-arvo, niinkö? *What does it say? Limit, or what?*
74 Pekka:		Ym. *Um.*
75 Mika:		Onko tää [epäselvää]raja-arvo. (Pekka koskettaa Mikaa olkapäähän ja viittaa omaan oppikirjaansa) *Is this [indistinct] a limit? (Pekka touches Mika on the shoulder and points to his text book.)*
76 Mika:		Misä täälä on [epäselvää]. (selaa taulukkokirjaansa) Tuosa, raja-arvoja. Kaaviossa. Joo. (naurahtaa) Mukavampiaki juttuja sitä ois. Lim x a kertaa log x. *Where do we have [indistinct]? (leafs through his formula booklet) Here, limits. In the diagram. Oh yeah. (gives a short laugh) Nicer things than this might exist. Lim x a times log x.*

Instead of following the questions on the worksheet and trying to construct knowledge by themselves, the students search for applicable information from their textbook and formula booklet. And they do not find what they are searching for. After a while the boys start expressing their own thinking and investigating mathematics and they successfully answer the questions. But when it is time to write a summary it is not clear to them whether they are expected to write down something, or to construct a formula. Instead of writing on the basis of their own thinking, the boys go and search for information from the textbook again.

An interesting, topic-specific phenomenon, related to the previous ones was realized in both of the groups. We studied the basics of the concept of derivative by investigating the movement of a glider on an air track. The students were asked to find the instantaneous velocity of the glider at one second after the start. The discussion about movement; distances and velocities triggered the context of physics for the students and they tried to remember principles from that field and apply them, instead of investigating by themselves and trying to find connections between the mathematical objects at hand. In the contemporary syllabus for physics, the method for finding the instantaneous velocity as the gradient of a tangent was taught in the first course to all first year students. Naturally the limiting process was not emphasized at that stage. But the students in my data did not remember the method.

4.2.2 Social norms produced or negotiated through the participation structures: peer interactions

Lampert (1990) emphasizes, with researchers in the interactionist tradition (Voigt, 1995), that to change conventional beliefs about what it means to know mathematics, teachers and students need to do different kinds of activities with different kinds of roles and responsibilities. By giving investigations to the students to work with I initiated, in an implicit way, the negotiation of the norm: *The role of the students in the investigative approach is to express **their own thinking** and to create new mathematics.* Through acting according to my expectations the students negotiated the same norm.

The negotiation was not, however, consistent. In episode 7 of the previous section the students in group A preferred to follow my instructions from the previous lesson instead of relying on their own observations and reasoning. The students acted according to the traditional norm: *The role of the students is to solve tasks according to instructions given by the teacher.* Yackel and Rasmussen (2002) describe expectations compatible with the school mathematics tradition and university mathematics tradition in the USA. The students participating in their experiment had earlier been expected to follow instructions and to solve problems in the way that the instructor or textbook demonstrated. This seems to be a similar situation to the school mathematics tradition in Finland. In episode 7, sticking to the old norm prohibited the students from learning about the concept of a limit in a new way.

By searching knowledge and statements from the textbook, instead of thinking for themselves, the students in episode 8 acted according to the traditional norm: *The role of the students is to apply ready-made knowledge.* According to Yackel and Rasmussen (2002) this contrasts with the expectations and obligations that underpin inquiry instruction where each student is expected to develop personally meaningful solutions for problems.

4.2.3 Participation structures in terms of knowing and learning: teacher-students interactions

When discussing with the students in the small-groups, I tried to act in a way compatible with the investigative approach. From my diaries during the experimental course it can be concluded that I realized that the traditional roles of the students and teacher have to be revised. But I was not always acting according to my new beliefs. The investigative approach challenged my teaching practice.

Episode 9

The students in group A had been investigating what happens to the values of the function $f(x) = \frac{\sin x}{x}$ (graph 2) when x approaches zero from the right and the left. There had been confusion among the students about how the x-values should be changing. An answer was given contrary to their own thinking. I joined the group and by reading Anni's notes I tried to get an idea of what the students had been doing.

198 Opettaja: Yhm. Mitäs täälä sitte (lukee Annin vihkosta)
Teacher: Um. How are things here? (reads from Anni's notebook)

199 Veikko: Ku me ei oikein ä, älytty, että, että ko ainahan ajatellaan että meiän pitää täälta vasemmalta kattoa ja tulla oikealle.
Well, we did not und, understand that, coz we always think that we have to come from here, from the left, and go right.

200 Opettaja: Ym.
Um.

201 Veikko: Niin sillonhan tässä on nouseva, tai kasvava. Ja tässähän se on sitte vähenevä (osoittaa kynällään Annin kuvaajasta).
So, in that case it rises here, or it is increasing. And, here it is, then, decreasing (points with his pen to the graph on Anni's calculator).

202 Opettaja: Ym. Ym.
Um. Um.

203 Veikko: Muttako tässä tehtävässä sanotaan että ku, että lähestytään nollaa oikealta. Niin periaatteessa se tarkottaa, että me mennäänkin tähän suuntaan (kuljettaa kynäänsä Annin kuvaajalla oikealta vasemmalle nollaa kohden).
But in this task it says that: when we approach zero from the right. So, basically, it means that we go to this direction, instead (follows with his pen the graph on Anni's calculator from right to left towards zero).

204 Opettaja: Juu, hyvä.
Yes, good!

205 Veikko: Mutta onko se sillon kasvava vai vain laskeva. Ku me ei oikein osattu päättää sitä [epäselvää].
But is it then increasing or decreasing? Because we couldn't decide [indistinct].

206 Opettaja: Joo. Nyt sillonku me puhutaan kasvamisesta, sillon liikutaan aina vasemmalta oikealle (siirtää sormeaan oppilaiden vasemmalta oikealle).
Yes. Now, when talking about the increasing of a function, then we always move from left to right (moves her finger from left to right).

207 Veikko: Joo.
Yes.

208 Opettaja: Ja nyt raja-arvotarkasteluissa me tehhään sillälai, että meillä on tietty x:n arvo, tuo nolla.
But now, when investigating limits, we do the following: we have a certain x-value, this zero.

209 Veikko: Ym.
Um.

210 Opettaja: Ja nyt lähestytään sitä, sekä oikealta (näyttää samalla sormellaan Annin kuaajasta) että täältä toiselta puolelta, vasemmalta. Ja mietitään mitä tapahtuu funktion arvoille.
And we approach that, from both the right (follows with her finger the graph) and from the other direction, from the left. And we consider what happens to the values of the function.

211 Veikko: No niin, okei.
Well, okay.

211 Anni: Eli onko se kasvava sitte siinä.
So is it increasing, then, there?

212 Veikko: Sit se on kasvava. (ryhtyvät kirjoittamaan ylös)
Then it is increasing. (both start writing down)

213 Opettaja: Funktion arvot? Niinkö? Tuosta kuvaajasta päättelitte. Vai? (opiskelijat keskeyttävät kirjoittamisen)
The values of the function, you mean? You saw that from the graph, or? (the students stop writing)

214 Anni: Niin.
Yes.

215 Veikko: Niin.
Yes.

216 Opettaja: Joo. Oikein hyvä. Tuota tehkääpä sitten numeerisia laskelmia. Tehkää vaikka semmonen taulukko johon pistätte x:n arvoiksi näitä (osoittaa x:n arvojen jonoja Annin vihkossa). Ja laskette niitä funktion arvoja. Niin sitte niistä lasketuista arvoista vois sitten vielä sanoa jotakin tarkemminkin, millä tavalla ne kasvavat.
Okay. Good. Well, then you could do some numerical calculations. Make a table, for example, with these x-values (points to the sequences of x-values in Anni's notebook). And evaluate the function values. Then from the function values you could say something more, in which way they increase.

In this episode, I am mostly trying to listen and understand the students' thinking. It leads me to explaining to them the difference between how x-values are changing when we investigate the increasing and decreasing functions and on the other hand when we try to find the limit of a function at a particular x-value. The students, or actually Veikko, led the course of the discussion. I believed that when investigating mathematics it should be the students' task to construct mathematics by themselves, and to express their thinking. There are episodes in my data where I acted in a way compatible with my beliefs. I listened to the students and gave space for them to express their ideas and questions. Sometimes I had to prompt the students to think for themselves or give them more time to arrive at their conclusions and occasionally I refused to answer a question and guided the students to doing further investigations by themselves.

But I was not consistent in my behavior. When I hear (211 – 213) the answer the students have developed for the question about what happens to the values of the function when x approaches zero from the right and from the left, I am astonished. The answer is not what I had expected. I wanted the students to see more; how the values of the function approach the number 1. My reaction reveals my thoughts and the students stop writing. I collect myself and remember that, when investigating, the role of the teacher should be to support students' own mathematical thinking and to find the important ideas by themselves. I decide to accept the students' answer (216), because after all it is not wrong and furthermore it can be developed. This episode reveals the struggle I had when trying to adjust my behavior to the demands of the new way of studying and to my beliefs about my own role and that of the students. The battle is, however, not yet won. The episode continues.

217 Veikko: Onko tässä periaatteessa niinku kaks suoraa? Pitääkö tuonne keskelle, nollan kohalle, piirtää vaikka joku ympyrä, että se ei niinku ikinä pääse siihen kuitenkaan. Miten se merkattas siihen?
Do we have in a way two lines here? Do we need to draw some kind of circle in the middle, at x-value zero? So that it never gets there. How should we mark it?

218 Opettaja: Miksi sie piirtäsit siihen ympyrän?
Teacher: *Why would you like to draw a circle there?*

219 Veikko: Eikö siis ko ne ei ikinä pääse siihen nollaan kuitenkaan. Ko ne vaan aina pienenee ja pienenee, mutta ne ei ikinä pääse sinne nollaan.
Well, because they never get to the zero, anyway. They only get smaller and smaller forever, but they never get to zero.

220 Opettaja: Joo-o.
Yea-ah.

221 Veikko: Eli
So

222 Opettaja: Tuota
Well

223 Veikko: voiko sitä merkata jotenki?
can you mark it somehow?

224 Opettaja: se, ää. Tuota tuohon pitäs periaatteessa piirtää aukko tuohon, mutta ei tästä syystä, vaan mistä syystä?
it, um. We should actually draw a circle there, but not for that reason. For what reason?

225 Veikko: [epäselvää] raja-arvo. Se ei niinku mee siihen (nauraa, ja kaikki muutkin nauravat)
[indistinct] limit. It kind of does not go there (laughs, and so do all the others).

226 Opettaja: Nii-i.
So-o.

227 Veikko: Yhm. Miten sen nyt sanos siinä
Um. How could you put it

228 Opettaja: Siinä, periaatteessa siinä pitäs siinä tuossa kuvajalla olla aukko tuossa kohtaa. Mutta miksi?
There, in principle there should be a hole on the graph at that x-value. But why?

229 Veikko: En sano mitään ko mie oon koko ajan äänessä (kaikki naurahtavat).
I am not going to say anything because I am speaking all the time (every one laughs).

230 Anni: Nii-hi.
Yeah-ha. (half laughing)

231 Opettaja: Katotaans tuota funktion lauseketta. … Mikä ongelma siinä nollassa on?
Let's have a look at the expression of the function. ... What problem do we have at zero?

232 Veikko: Niinkö nollalla ei oo arvoa, nollalla ei voi jakkaa.
It doesn't have a value at zero, you cannot divide by zero.

233 Anni: Niin.
Yes.

234 Opettaja: Tuota funktiota, sillä ei oo arvoa siinä nollassa. Siksi siihen voi panna semmosen avoimen ympyrän. Ja se on samanlainen tilanne minkä sää niinkö siinä selitit äsken. Mutta tuota tehkääpä semmonen taulukko, jossa on ensin nuita x:n arvoja ja sitten funktion arvoja laskettuna (piirtää pöytään sormellaan kuin taulukon). Niin sillon te voitte sanoa tarkemmin tuosta millä lailla ne kasvaa.
Well, the function, it hasn't got a value at zero. That is why we should draw such an open circle. And the situation is similar as you just described. But, well, make a table with x-values and the evaluated function values (draws with her finger a table on the surface of the desk). Then you could say something more exact about how they increase.

235 Veikko: Juu. (opettaja poistuu)
Ok. (the teacher exits the room)

In this latter part of the episode I still try to continue with my new philosophy (218), but I had lost my inventive power and I started playing a “guessing game” (224 – 235), where the idea is that the students have to find out what the teacher has in mind and try to answer so that she is pleased. Sometimes it seems to be hard for me to discuss with my students in a genuine way. In many episodes in the two small groups I am putting forward my own agenda at the cost of not really listening to the students’ thinking and of not communicating with them in an authentic way. In many of those occasions, I could have started my discussion more closely from the students’ ideas.

The assertiveness of the three boys, Veikko in group A and Mika and Juha in group B, had also positive communicative consequences. Through their behavior I was forced to see the perspectives and genuine questions of the students. In the following episode the students in group B have been investigating the connections between a function and its derivative function (problem 6). Mika has an important question and he takes the floor to express it and leads the discussion for the most part.

Episode 10

1	Mika:	Hei ope! *Hey teacher!*
2	Opettaja: *Teacher:*	Niin. *Yes.*
3	Mika:	(osoittaa kynällään kysymyspaperin toista kysymystä) Tässäkö täsä kysytään tätä derivaattaa, niin onko tää niinkö derivaattafunktio. Ko täsä on, eikö tää oo muuten ihan sama asia? (osoittaa ensimmäistä kysymystä, opettaja laskeutuu oppilaiden tasolle, nojaamaan kyynärpäillään pöytään) *(points with his pen to the second question in the assignment) There’s this question about the derivative, so, well, does it concern the derivative function? Because here we have, isn’t it by the way exactly the same question? (points to the first question, the teacher crouches down at the level of the students)*
4	Opettaja:	Aa. Mitä se derivaatta liittyy nuihin tangentteihin (osoittaa sormellaan kysymyspaperia)? *Um. How is the derivative connected to those tangents (points with her finger to the question paper)?*
5	Mika:	No derivaattahan oli se tangentin kulmakerroin. *Well, the derivative is the gradient of the tangent.*
6	Opettaja:	Nii. Eli nyt tuota *Yeah. Well then*
7	Mika:	Eikse oo ihan niinkö sama kysymys (osoittaa kynällään kysymyspaperia)? *Isn’t it exactly the same question (points with his pen to the question sheet)?*

8 Opettaja: No, no ei aivan ko tangentti on. Tangentti on tangentti, se on suora (osoittaa kädellään pöytään kuin suoraa).
Well, not quite, because a tangent is. Tangent is a tangent, it is a line (shows with her hand like a line).

9 Mika: Ym.
Yes.

10 Opettaja: Ja sitten tangentin kulmakerroin on taas luku (katsoo Mikain ja Juhain). Tavallaan, että jos nyt aivan. Mutta nehän on
And then the gradient of the tangent is a number (looks at Mika and Juha). If you put it exactly. But nevertheless they are

11 Mika: Niin, mutta eihän nyt tuosta tangentista mittään muuta oo hyötyä ollu paitsi se, se kulmakerroin.
Yes, but the tangent was of no use except its gradient.

12 Opettaja: No niin, ei. Ei.
Well, no. No.

13 Mika: Niin.
Yes, so.

14 Opettaja: Mutta mitäs te vastasitte tuohon ensimmäiseen kysymykseen?
But, what did you answer to the first question?

15 Mika: Me vastattiin, että sillonko se tangentti, se kulmakerroin, on suurempi ku nolla, niin se on kasvava se, siinä kohasa se, se, se yht, se kuvaaja.
We answered that when the tangent, the gradient, is greater than zero, it increases there the equat, the graph.

16 Opettaja: Joo. Funktio.
Yes, the function.

17 Mika: Funktio, niin.
The function, yes.

18 Opettaja: Joo.
Yes.

19 Mika: Ja sittekö se on pienempi, niin se on laskeva.
And when it is smaller, then it is decreasing.

20 Opettaja: Kyllä. Eli te ootte ite asiassa pohtinu nämä kaks asiaa samalla kertaa (näyttää sormillaan kysymyspaperin kahta ensimmäistä kysymystä).
Yes. So as a matter of fact you have been considering these two questions at the same time (points with her finger to the first two questions on the sheet).

21 Mika: Niinku meistä se on sama asia.
Well, because for us it's the same thing.

22 Opettaja: Joo. No ni. Hyvä.
Ok. So. Well done!

23	Mika:	Ja sitten ku se on. Tai onko se niinku nolla tai sitä ei ole sitä kulmakerrointa, niin sillon se on niinkö vaakasuorasa. *And when it is. Is it zero or the gradient does not exist, then it is sort of horizontal.*
24	Opettaja:	(nyökkää) Sillon tangentti, siihen kohtaan piirretty, on vaakasuora. *(nods) Then the tangent, the tangent drawn at that point is horizontal.*
25	Mika:	Ym. *Um.*
26	Juha:	Ym. *Um.*
27	Mika:	Että se ei kasva eikä laske siinä. *That it does not increase nor decrease at that point.*

In the beginning, Mika is active and poses me a question. During turns 4 and 6 I start thinking about how I should start an appropriate instructional discussion. But in turns 7 and 11 Mika interrupts me and forces me to listen to his point of view and comment directly on that. Turn 11 is also an example of Mika challenging my interpretation. Mika really acts in a way compatible with the new role of the students. Even though in my experimental course I did not negotiate new social norms in an explicit way, there can be seen signs of beliefs compatible with the investigative approach, similar beliefs as in the experimental inquiry mathematics course of Yackel and Rasmussen (2002), for example.

4.2.4 Social norms negotiated or produced through the participation structures: teacher-students interactions

When my students expressed their own questions and comments and when I listened to them and proved to communicate with them in a genuine way, we were negotiating in an implicit way the norm: *In the investigative small-group approach the role of the students is to express **their own thinking** and to create mathematics.* But my negotiation of social norms was contradictory. Many times, for one reason or another, I only wanted to realize my own agenda for meeting the group and keeping them on the "right track". In those situations I expressed my belief compatible with the traditional norm: *It is the role of the students to find out the official interpretations and meanings of mathematical objects.*

When failing in adjusting my behaviors to the investigative approach, I acted like two teachers in Pang's research in Korea. In the first report Pang (2001) describes how the teacher usually listened to students' various contributions but finally turned the classroom discussion toward one direction, to a standard algorithm or one specific equation. If the students did not arrive, for example, at the prescribed form of equations, the teacher introduced it even after students' reasonable thinking. In the second paper Pang (2005) shows how the teacher made a promising transition toward successful student-centered teaching practices. But in those cases when the students did not arrive at an idea she had in mind, the teacher provided the students with crucial hints that changed the nature of the activity from inventing to following the strategy the teacher had in mind.

For me there is a twofold struggle here. On the one hand I was tied to my routines from traditional teaching. Acting in a different way as a teacher is not just a matter of changing your beliefs. You also have to find out how to do things differently in action. On the other hand it is especially difficult in a situation of an overloaded normative syllabus and central examinations. Mathematics teaching in Finnish upper secondary schools aims at the acquisition of standard procedures and concepts, even specific ways of symbolizing and presenting solutions. In such a context it is very difficult for the teacher to let go and give space for students' own authentic thinking.

Richardson and Placier (2001) summarize research on teacher change. According to them the change from traditional teaching to applying constructivist principles in instruction seems to be difficult for many teachers and student teachers. Deep and lasting change requires a consideration of a multitude of aspects and interests. It should be viewed as an ongoing process. Most successful in changing teachers' beliefs, conceptions and practices in this direction have been long-term, collaborative and inquiry-oriented programs (Richardson and Placier, 2001, Cobb, Wood and Yackel, 1990).

Wood, Cobb and Yackel (1991) describe the learning of a teacher in their second grade experimental classroom. During the experiment the teacher went through three major reorganizations in her beliefs about a) the role of a teacher from transmitting information to supporting the construction of students' mathematical meanings b) the role of the teacher from imposing her own methods to negotiation of meaning with the students and c) the role of the teacher from mere encourager to act through the negotiations as an integral part of the learning environment. It is easy for me to understand the teacher in this experiment. At the time of my experimental course I seem to have gone through the first reorganization at least to a certain extent. It was the motivation for me to teach the basics of calculus through the investigative small-group approach with the emphasis on concepts and meaning construction. Evidence of it can also be seen in the data. During the experimental course I was struggling with the second aspect. The teacher in Wood, Cobb and Yackel's experiment tried to resolve the conflict inherent in the second reorganization by a discourse pattern called "funneling" (Bauersfled, 1980) where she, by revealing the solution bit by bit, tried to ensure that the students get the right answer whether they understand it or not. In the discussions with the research group the teacher discovered that in this way she was actually depriving the children of the chance to think through the solution for them selves. At present I am still working with aspects b and c in my use of the investigative approach.

4.3 Summary of the social norms

Collaboration and communication

In both of the groups the actions of the students conveyed the existence and production of the collaboration norm:

- When working in small-groups, students are expected to try to arrive to a consensual conclusion.

When collaborating, the students in both groups produced parallels to some basic social norms for inquiry mathematics:

- When working in small-groups, students are expected to **express** their own thinking.
- In small-group discussions students are expected to listen to the others.

However, in both of the groups and in teacher-students interactions the right to speak and the obligation to listen were unevenly distributed. During the moments when this principle was realized, we were producing the norm:

- Those who are more assertive than others have the right to express their thinking, and others have the obligation to listen to them.

A striking phenomenon in the data was that we were not accustomed to justifying our mathematical claims. In student-student interactions as well as in the discussions between me and the students we accepted social rather than mathematical reasons for our claims. Production of two norms was connected to this:

- The acceptance of a mathematical claim is on the basis of social agreement.
- The acceptance of a mathematical claim is on the authority of the speaker.

An exception to this trend was that in the peer interaction of group B the students produced, to a greater extent than in other interactional make-ups, the norm:

- In small-group discussions students should justify their claims.

In group B the students produced the norms:

- When working in small-groups, students are expected to show agreement.
- In small-group discussions students are expected to show disagreement and challenge each other.

On the other hand, in group A, one could see the production of the following norms:

- When working in small-groups, students are expected to show agreement.
- In small-group discussions it is not appropriate to show disagreement too clearly.

In addition, in group B, two special norms and their production could be observed:

- In small-group activities the role of the person with mathematical competence is to listen to those who have problems and help them.
- A student should show competence and confidence in mathematics.

Knowing and learning

In terms of knowing and learning I initiated (in an implicit way) the negotiation of the norm and the students took the challenge and produced the norm:

- In the investigative approach the role of the students is to express their **own thinking** and create new mathematics.

My negotiation, however, was not consistent. Every now and then I acted according to and thus produced a norm which worked against the philosophy of the investigative approach:

- It is the role of the students to find the official interpretations and meanings of mathematical objects.

Also my students were sometimes acting according to the traditional norms:

- The role of the students is to solve tasks according to instructions given by the teacher.
- The role of the students is to apply ready-made knowledge.

Most of the social norms produced or negotiated here can also be found in the research literature. However, some regularities, that I have interpreted as normative behaviors I could not find classified as norms elsewhere. There was a clear regularity in the ways of speaking and listening in our interactions. Those who were more assertive, who took the floor for themselves, had the right to speak and others had the obligation to listen. Our discussions were by no means democratic so that if anybody had had a relevant mathematical argument she would have had the possibility of expressing it. This phenomenon is recognized in many analyses of discussions and interactions, but I have not seen it classified as a social norm. The right of assertive persons to speak and the obligation of others to listen was a regularity, or pattern of interaction, mutually constituted in the interactions of the small-groups. It described the way my students and I were participating in the communication of the small-groups. According to my theoretical framework it can be seen as a social norm.

In group B the boys were producing a norm about the role of the more competent student to assist the low-achievers. The students and teacher in the interviews of Mulryan's unpublished dissertation (Good et al., 1992) mentioned help-giving/help-seeking as expected behavior in collaborative small-groups. They saw it as normative behavior, although the concept of norm was not discussed by Mulryan.

The conduct of the boys in group B of presenting their competence in mathematics very often may be a feature typical of boy's sociolinguistic subculture, the features of which are a close concept to a social norm. In this case it describes the way the boys were participating in the communication of their small-group. For this reason I have classified it as a social norm.

5 SOCIOMATHEMATICAL NORMS

The second research question was: What kind of sociomathematical norms were produced or negotiated in the interactions of the two small-groups. This chapter aims to answering that question.

5.1 A mathematical investigation should be approached in a profound and creative way

At the beginning of the course my students were not accustomed to approaching an investigation. In their experience solving mathematical tasks had normally meant following certain given procedures. A required answer to a mathematical task had usually been one number or expression, or sometimes a simple answer given by a word or two. In that respect, I think, the situation in my class was similar to the traditional didactic contract in Danish mathematics lessons described by Blomhöj (1994). Before the experimental course my students had not, except on some rare occasions, been asked to do observations and to describe phenomena in mathematics.

Episode 1

When investigating the limit of the function $\frac{\sin x}{x}$ at x = 0, the students in group A first gave an answer that when x approaches zero from the right or from the left the values of the function increase. I was not satisfied with their vague answer and I had to prompt them to observe more exactly.

216 Opettaja: Joo. Oikein hyvä. Tuota tehkääpä sitten numeerisia laskelmia. Tehkää vaikka semmonen taulukko johon pistätte x:n arvoiksi näitä (osoittaa x:n arvojen jonoja Annin vihkossa). Ja laskette niitä funktion arvoja. Niin sitte niistä lasketuista arvoista vois sitten vielä sanoa jotakin tarkemminkin, millä tavalla ne kasvavat.
Okay. Good! Well, then you could do some numerical calculations. Make a table, for example, with these x-values (points to the sequences of x-values in Anni's notebook). And evaluate the function values. From the function values, then, you could say something more about the way how they increase.

At the end of the session Anni constructs, and the others agree, that the values of the function get closer and closer to the number one when x approaches zero from the right or from the left. But when writing their summary, the students start talking only about the increasing function. Again, I have to ask them about the specific way the values of the function increase.

When working with the second investigation, where the values of the function increase to infinity or decrease to minus infinity when x approaches a certain number, the students give some attention to how the values of the function increase or decrease. But in their notes they only write about the increasing and decreasing of the values of the function. When I come to see the group, I read the notes and ask the students to be more exact.

142 Opettaja: Osaatteko sanoa siitä millä tavalla ne kasvavat? Niinku sillon kerran ne kasvoivat lähestyen ykköstä, niin voisko nyt sanoa enemmän millä tavalla ne kasvaa?
Teacher: Could you tell about how they increase? Like once they increased by approaching one, so could you say something more about the way how they increase.

143 Anni: Silleen (naurahtaa, näyttää kynällään kuinka ensimmäisen funktion arvot menevät äärettömään kun x lähestyy miinus ykköstä). Miten siis, mikä?
So that (laughs, shows by her pen how the values of the first function approach infinity when x approaches minus one). How, what?

144 Veikko: Siis räjähdysmäisesti, niinkö?
Well, in an explosive way, or?

145 Opettaja: Joo.
Ok.

During the third investigation about limits at infinity and at minus infinity, the students first talked only about increasing and decreasing functions, but soon, without my prompting, they continued by investigating the specific ways of how the values of the function increase or decrease. And they found, for example, that the values of the function $\frac{x^2}{x^2-1}$ "don't decrease below 1" and constructed that there might be a limit parallel to the x-axis.

During these interactions with my students, we negotiated the sociomathematical norm: *When investigating mathematics, one should try to approach the topic in a profound and creative way.* Too simple and repetitive answers are not approved.

I have not met descriptions of this kind of norm in the literature. It may be because of the differences between my approach and realistic mathematics education. However, according to my experiences about the use of the investigative approach, the problem of superficial processes and simple answers has often shown up. Also, many research reports on the use of small-groups in instruction suggest that communication between students in peer interaction may be shallow such that it does not foster meaning (Kumpulainen, 2002, Roth and Roychoudhury, 1992). Lampert (1990) argues that, in popular culture and in most classrooms in the USA, mathematics is associated with certainty. What is important in knowing it, is to be able to get the right answer and quickly. Mathematics is not a subject to be explored and created (Lampert, 1990.) In their review of the use of small-groups in instruction, Good et al. (1992) call for willingness of students to do explorations and to ponder as well as to try to resist the temptation to go faster. Thus, it may be anticipated that the negotiation of this norm is an important aspect of developing the use of the investigative small-group approach.

The sociomathematicl norm about profoundness and creativeness overlaps to some extent with the social norm, which is connected to knowing and learning: *In the investigative approach the role of the students is to express their own thinking and create new mathematics*. But I want to include it as a sociomathematical norm as well, because here it is connected specifically to the way that a *mathematical* task is approached.

5.2 Different ways are accepted

The assessment policy of the national mathematics examination board in Finland is such that we upper secondary school teachers have to guide our students to follow certain rules about accepted ways of solving mathematical tasks and the form and accuracy of answers. In the first small-group session in group A, the freedom of an investigation seemed to cause trouble for the students. The first discussion with me is illuminating.

Episode 2

In the introduction to the first small-group session (problem 1) I had shown the students two infinite number sequences. I had written them down on a transparency so that each sequence started from a number, and the terms of the sequence were separated by commas. At the beginning of their small-group session, Veikko had used his calculator to take some x-values and function values for the investigated function $\frac{\sin x}{x}$. But Anni had been insisting that they should write down similar sequences as I had shown them.

85 Anni: (opettajalle) Hei miten tää homma oli, silleenkö meillä meni silleen
(to the teacher) So, how was this task? In a way we did not

86 Veikko: Eli tuo lukujono, miten se valmistetaan siis silleen niinkö että. Ko me kyllä, mie kyllä tajusin siellä luokassa et ku oli puoli ja sitte tulee yksi ja kaks.
We mean this sequence, well, how do we construct it? Because we did, I did understand in the class that when you have half and then comes one and two.

87 Anni: Niin pitäskö tää (osoittaa kynällään kysymystä tehtäväpaperissa) olla niinku samantyylinen?
So, should this (points with her pen to the question on the worksheet) be somehow similar?

88 Opettaja: Sam, joo-o, samantyylinen jollakin, jonkin sortin idealla. Sillä lailla, (osoittaa sormellaan kysymystä Annin tehtäväpaperissa) että sen luvut on koko ajan nollaa suurempia ja ne niinkö lähestyvät nollaa. Ja se on vielä semmonen, että se on päättymätön. Eli että te voitte tietää, että kaikki ne jonon jäsenet on nollaa suurempia ja aina lähempänä ja lähempänä nollaa.
Teacher: Sim, yeah, similar with some, some kind of idea. So that (points with her finger to the question on Anni's worksheet) the numbers are all the time greater than zero but they approach zero. And it is also infinite, so that it never ends. So that you can know that all the terms in the sequence are greater than zero and always closer and closer to zero.

89 Veikko: (naurahtaa) Okei.
(makes a short laugh) Okay.

90 Opettaja: Lähe, lähtekääpä jostakin liikkeelle.
Start, start with something.

91 Veikko: Mut miten se niinku pittää merkata sitte?
But how should we write it down then?

92 Opettaja: Vaikka silleen, että pilkulla erotatte niitä lukuja.
Well, for example so that you separate the numbers by commas.

93 Veikko: Okei. Niin justiinsa.
Okay. Yes.

94 Opettaja: Ja jatkatte tarpeeksi pitkälle.
And you should continue far enough.

95 Veikko: Ja kuinka monta desimaalia me sinne laitetaan?
And how many decimal places should we write down

96 Opettaja: Se riippuu vähän että minkälaisen jonon te laitatte, et minkälaisia arvoja te laitatte. Koettakaas miettiä tuo ensin (osoittaa sormellaan kysymystä Annin monisteessa) niin, niin tuota se auttaa teitä ymmärtämään sitte niinkö etteenpäin. Koettakaa muodostaa tuommoinen lukujono.
It depends a little bit on what kind of sequence you construct, on what kind of values you choose. Try to think about this first (points with her finger to a question on Anni's worksheet), then, it may help you to understand more. Try to construct that kind of sequence.

97 Anni: Mut siis, tarviiko sitä nyt mitenkään laskimesta kattoa vai miettiä vaan miten ne on?
But, well, do we need to find it from the calculator or just think about such numbers?

98 Opettaja: Ne voi ihan minusta miettiä niinkö päässä, että, että voihan niitä laskimella tietenki sievennellä ja muuta, että ihan omalla ajatuksella.
I see that you can decide them without a calculator, that, that of course you can simplify them with the calculator and such, but now you can also use your own brains.

99 Veikko: Siis, elikkä meiän ei tarvi välttämättä käyttää tätä kuvaajaa, joka tästä tulee. Voiaanko me muodostaa se vaikka että, niin, no joo.
So we don't necessarily need to use this graph, which the calculator draws. Can we make it for example so that, well, okay.

100 Opettaja: Niin tää x:n arvojen jono voidaan tehdä ihan, niinku ite päättää, että mitkä luvut.
Well, this sequence of x-values can be constructed so that you decide which numbers you choose.

When I come to visit the group, the students start asking me whether the sequences they are supposed to construct should be similar to my sequences (85 – 87). Veikko is further wondering how they should write down the sequences (91) and he asks for the number of decimals they should use (95). Then Anni asks their original question more clearly (97): Should the students take the sequence from the calculator or just decide the numbers. In an unusual situation the students act according to the traditional sociomathematical norm in our school: *There are certain rules for solution methods and for writing down the solution and the answer of a mathematical task.* I am answering their questions, but the way I talk conveys my belief that, in this case, these details are not so important. I answer Veikko and Anni (92) "Well, *for example* so that", (96) "*It depends a little bit on* what kind of a sequence you construct." and (98) "*I see* that you can decide". I am implicitly giving the message that now there are no universal rules for method and writing down the solution and answer. While I am helping the students and giving answers to their questions, I am at the same time expressing my belief compatible with the sociomathematicial norm: *When* *investigating mathematics, the questions about method, accuracy and writing down the solution are situation dependent.*

As the other group was working with the same investigation, Mika had difficulties in accepting the concrete style of the task.

Episode 3

When the boys finally decided to start working with the questions, instead of searching for knowledge from the textbooks, Pekka and Mika discussed how to construct a required sequence, the numbers of which approach zero from the right.

96 Pekka: Niin. Pistetään funktio (näppäilee laskintaan).
Well. Let's write a function (taps his calculator).

97 Mika: Niin, eli minkälainen kaava siihen saahan? (Reijo kohauttaa hartioitaan) Ettei ko, koskaan niinku tuu (piirtää sormellaan ilmaan kuin funktion kuvaajaa) vhiuh, nouseva.
So, then, what kind of formula can we make? (Reijo shrugs his shoulders) So that it never, it never becomes (draws with his finger in the air like a graph of a function) wiuuuh, increasing.

98 Pekka: Katotaanpa (näppäilee laskintaan).
Let's see. (taps his calculator)

99 Mika: Oisko joku, yks jaettuna, yks jaettuna x:l, jotaki. Piirtääkse mittään sulle siitä? (katsoo Reijoun)
Could it be something, one divided, one divided by x, or something. Does it draw anything for you? (looks at Reijo)

After a while the boys continue the discussion.

104 Mika: No mie keksin heti tomm, tuommosen.
Well, I got it already, like, like a.

105 Pekka: No?
So?

106 Mika: Jono, eihän se oo mikään yhtälö. Mie laitan jotenki, että yks jaettuna kymmenen (kirjoittaa vihkoonsa), yks jaettuna sata, yks jaettuna tuhat. Eikö siis miten se oli? Yks jaettuna, väärin päin meni (pyyhkii pois). Ootappa (näppäilee laskintaan). On, on se niin. Yks jaettuna tuhat. Niin, siinä se on.
Sequence, it's not an equation. I'll put something like one divided by ten (writes in his notebook), one divided by a hundred, one divided by a thousand. Well, how is it? One divided by, oh no, it's wrong (erases something from his notebook). Wait a minute (taps his calculator). Yes so it is. One divided by a thousand. Yes, there we are.

107 Pekka: (valmiina kirjoittamaan) Niin mitä?
(ready for writing) So, what?

108 Mika: Ni. Tuosta se tullee. Ne nollaa lähenee koko ajan.
Yes. This is it. They approach zero all the time.

109 Pekka: [epäselvää] x:llä.
[indistinct] by x.

110 Mika: Eiku yks Niin.
No, one Yes.

111 Pekka: Yks jaettuna
One divided by

112 Mika: Yks jaettuna kymmenen, yks jaettuna sata, yks jaettuna tuhat ja niin etteespäin. (Pekka ja Reijokin kirjoittavat lukuja ylös)
One divided by ten, one divided by a hundred, one divided by a thousand and so on. (also Pekka and Reijo write down the numbers)

Together Pekka and Mika start searching for a symbolic solution (96 – 99). But after a while Mika constructs a concrete sequence (106) and even gives a justification for it (108). The other boys accept his solution. At the end of the investigation the students discuss how to write a summary.

313 Mika: Minusta tuntuu, että tämä ei taia mennä oikein, tämä juttu.
I have got a feeling that it's not going right, this thing.

314 Reijo: Miten niin?
Why do you think that?

315 Mika: Siis tuo, pitäskö tässä jotaki kehittää jotaki yhteenvetoa, niin mitä se yhteenveto on? Onko se jotaki kaavaa vai?
Well, should we develop some kind of summary, but what is it? Is it some kind of formula, or what?

316 Reijo: Mitä sie oikein selevität?
What are you talking about?

317 Mika: Jos mie kirijotan tuota yhteenvetoa.
If I am writing the summary.

318 Pekka: Niin.
Yes.

319 Mika: Ni.
Yes.

320 Pekka: Yhteenveto.
Summary.

321 Mika: Ni.
Yes.

322 Pekka: Jos sie teet yhteenvetoa
If you are writing a summary

323 Mika: Niin mitä me siihen kirijotettaan?
So, what shall we write there?

324 Pekka:	Jos sie teet yhteenvetoa jalakapallokisoista, niin et sie siihen ala kaavaa kirijottamaan. *If you are making a summary of a football match, you are not going to do it by writing a formula.*
325 Reijo:	Mitä [epäselvää]? *What [indistinct]?*
326 Mika:	Kaavajuttu. Mie kirijotan jonkinnäkösen kaavan. (Reijo ja Pekka naurahtavat) *Formula thing. I'll write some kind of formula. (Reijo and Pekka laugh a bit)*
327 Reijo:	Oho. (naureskelee) [epäselvää] tietää kaikki. *Well, well, well. (laughs) [indistinct] knows everything.*
328 Mika:	Aivan sama. *All the same.*
329 Pekka:	No se pittää vain kirijottaa. Se on niinkö (työskentelee laskimellaan, samoin muut pojat) *Well, you just have to write it. Really (works with his calculator, like the other boys)*

In this episode Mika is often trying to fulfill the expectations of teachers, textbook writers and the examination board for high level students. He tries to act according to the sociomathematical norm: *An accepted solution for a mathematical task must be in symbolic form.* Normally numerical or graphical solutions are not accepted during lessons or in exams, let alone stories. Pekka is more ready to use his common sense and suggests that they should actually write the summary in words. In the end of the session Mika asked me whether the summary should be some kind of formula. I looked at a written summary in Pekka's notebook and told the boys that they already had a nice summary. I thus expressed my belief which conflicts the normative appreciation of symbolic representation when investigating mathematics.

This sociomathematical norm is a special case of the norm discussed on the basis of episode 2. However, I find that it deserves special attention because it was so clearly seen in the data and because it works against the possibilities of fully utilizing the investigative approach.

Edwards (2007) followed the development of sociomathemtical norms in friendship groups of 14 – 15 year old students in UK. She shows how low attaining year 10 girls were refining their conception of mathematical efficiency by expressing their idea in a symbolic form. She suggests that the norm of appreciation of the symbolic representation is not mentioned in Cobb and Yackel's work, perhaps because their experiments were done mostly in first and second grades. Although I saw signs of the same norm, the situation of Edwards' students was different from that of mine. Her students were doing the process of symbolizing by themselves, and thus making their contributions more efficient.

In my episode Mika had problems in going to the concrete at all. I have the anticipation that, in traditional teaching in Finnish upper secondary schools, the overt appreciation of symbolic representation and methods very often makes the manipulation of symbols meaningless activity. Nunes, Schliemann and Carraher (1993) refer to the phenomenon by writing that "school mathematics represents a syntactic approach, according to which a set of rules for operating on numbers is applied in problem solving. Meaning is set aside for the sake of generality" (p. 103). In constructing mathematical meaning, we often need to deviate from strict adherence to the appreciation of symbolic methods only.

The symbolic calculator made it possible to use different approaches to the questions and problems posed to the students. During the experimental course I encouraged the students to utilize the graphic, numerical and symbolic functions of the calculator. And they approached the questions in different ways, with the calculator and without it. Through these interactions and actions we were negotiating the norm: W*hen investigating mathematics different approaches, in addition to symbolic methods are approved*.

From a psychological constructivist point of view Repo (1996) argues that to support conceptual learning and reflective abstraction in a CAS (computer algebra systems) environment calculus class, students should be encouraged to use different representations like symbolic, graphical and numerical. Ju and Kwon (2007) describe that in their inquiry oriented differential equations class graphical, numerical and qualitative mathematical methods as well as analytic methods were integrated. And Park et al. (2007) report this principle as a sociomathematical norm describing the culture of an inquiry oriented differential equations class.

5.3 The nature of mathematical talk

Because of the type of tasks I gave to the students, there was a lot of talk about mathematical objects in the discussions of the two small groups. By this, I mean that the mathematical talk was not only about how to perform procedures or how to solve tasks. We talked about (experientially real) mathematical reality like the students in the following extract.

Graph 3 The graph of the function f(x) = $-\dfrac{x}{(x+1)^2}$

Episode 4

The students in group B were beginning to work with the second investigation about the limits infinity and minus infinity at a particular x-value (problem 2). The boys have drawn the graph of the function f(x) = . Mika has tried to evaluate with the calculator the value of the function at x = -1 and obtained the symbol ∞.

17	Reijo:	Ne lähestyy ääretöntä (katsoo kysyvästi Mikaa). *They approach infinity (looks at Mika with a questioning expression).*
18	Mika:	Niin se jatkuu. Niin. (Reijo nyökkää Juhalle) Aivan. *Yes, it continues. Yes. (Reijo nods to Juha) Exactly.*
19	Juha:	Aa, aa, mitä se sitte käytännössä niinkö tarkottaa? Sitä, sitä *Aa, aa, what does it mean in practice, then? That, that*
20	Reijo:	Funktion arvojen, funktion arvot lähestyvät ääretöntä. Se on se vastaus. *The values of the function, the values of the function approach infinity. That is the answer.*
21	Mika:	Ym. *Um.*

22 Mika: Ku on tämmönen ylöspäin, ylöspäin, ylöspäin.
When we have this kind of upwards, upwards, upwards.

23 Juha: Mut siis. Minkä näkönen siitä kuvaajasta sillon tulee? (selaa vihkoaan)
But. What is the graph like then? (leafing through his notebook)

24 Mika: No semmonen (viittaa kynällään Antin laskimeen ja sen jälkeen piirtää kynällään ilmaan kuin funktion kuvaajan nousun kohdassa x = -1).
Well, this kind of (points with his pen to Juha's calculator and draws with his pen in the air the graph which is increasing near x = -1).

25 Juha: Äh. En minä sitä tarkottanu, vaan siis. Anna olla. (viittaa kädellään Mikan päin)
No. I didn't mean that, but, let it be. (points with his hand at Mika)

26 Mika: Mitä nuo on? Miten se muuten lasku tulee täältä?
What are these? How does it, by the way, decrease from here?

27 Reijo: Hm?
What?

28 Juha: Sitä mie juuri tarkotin, että kohtaako ne sen viivan vasen ja oikea puoli ikinä?
That's what I meant: do the right and left sides of the line ever meet?

29 Reijo: Ei.
No.

30 Juha: Koska periaatteessa, jos siinä ois joku, niin siinä ois piste. (pistää kätensä asentoon, kuin kuvaaja)
Because in principle, if there were some, then there would be a point. (illustrates the graph with his hand)

31 Reijo: Ei ne kohtaa ko siinä ei oo sitä.
They don't meet there, because it does not exist.

32 Mika: Ym.
Yeah.

33 Reijo: Siinä ei oo sitä.
It does not exist there.

34 Mika: Duuum (liikuttaa käsitään, kuin ne kulkisivat kuvaajalla ylöspäin lähestyen kohtaa x = -1).
Duuum (moves his hands as if they were moving on the graph upwards and approaching x = -1).

35 Juha: Omituinen.
Strange.

36 Reijo: Miten niin?
Why?

37 Juha: No ihimiset on kehittäny matematiikan ja sitte siinä on tämmösiä aukkoja. (naurahtaa)
Well, people have constructed mathematics and it has these kind of holes. (laughs)

Although for a short while Reijo and Mika (20 – 22) refer to the aim of finding an answer, otherwise the discussion is about values of the function and about the graph. Through this kind of talk the students and I were producing the norm: *Mathematical discussion is about mathematical reality.*

However, we were not consistent in producing this norm. Our talk easily slipped into instructions about how to solve tasks and problems.

Episode 5

Group A was sketching the graph of a third order polynomial function starting from the graph of its derivative function (problem 6). Anni had been thinking for a while about the question of which height she should draw the graph.

95 Anni: (opettajalle) Onko se niinku ihan sama, että mille kohalle se on piirretty, että jos siinä on se idea?
(to the teacher) Well, is it all the same regardless of what height it is drawn at, if, however, the idea is right?

96 Opettaja: Joo tavallaan, että mille korkeudelle (näyttää kädellä eri korkeuksia), niin sehän on ihan sama.
Teacher: Yeah, in a way, the height (shows with her hand different heights) does not matter.

97 Anni: Niin.
Yes.

98 Opettaja: Kunhan siinä vain se kasvaminen ja väheneminen mennee oikeinpäin. Kyllä minusta näyttää, että teillä on oikein hyvä, hyvä ajatus tuossa. Joo-o.
If only the increasing and decreasing are right. It looks to me that you've got a very good idea here. Yes.

Anni and I dicsuss about a proper way of solving the task. We come to the conclusion: If the increasing and decreasing of the function are expressed right, then the task is properly solved. We were not talking about functions, their derivative functions and the fact that a function can be the gradient function of many others which have the same shape but are at different heights. It seems to me that Anni triggers the style of my answer by saying at the end of her first utterance (95) the words "if, however, the idea is right". There would have been a wonderful opportunity to discuss the mathematical situation and give a short introduction to integral functions. But the opportunity was lost. Through this kind of talk we were producing the norm: *Mathematical discussion is about solving tasks.*

Jiménez-Alexandre et al. (2000) report that in their study on the use of small-group discussions in science lessons, much of the talk in the small groups was about "doing the lesson" and not about the expected focus of the discussion.

McClain and Cobb (2001) describe the development of sociomathematical norms in a first-grade inquiry mathematics classroom. They show the emergence of the norm that an acceptable explanation has to carry the significance of actions on mathematical objects. Elsewhere, Yackel and Cobb (1996) refer to the same norm by writing that, because of the establishing that, the explanations of students were conceptual rather than calculational. In developing inquiry mathematics cultures, Cobb and Yackel (1998) say they have worked towards the ideal that the teacher and students together act in and elaborate a taken-as-shared mathematical reality. Argument in inquiry mathematics classroom establishes mathematical truths, and teacher's and students' negotiations are about the nature of an emerging mathematical reality. On the other hand, the practices in school mathematics classrooms can be called procedural instructions (Cobb and Yackel, 1998.) In line with previous researchers, Sfard and Linchevski (1994) argue that the metaphor of acting on mathematical objects involves a shift away from the view of mathematical activity as information processing toward that of acting in a virtual mathematical reality. Sfard (2000) further develops her thesis and explains how mathematical discourse and mathematical objects create each other. "It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions" (2000, p. 47). In the light of these writings, the importance of discussion about mathematical objects seems to be an inevitable feature of inquiry mathematics and essential for the conceptual learning of students.

I see that through producing the two contradicting norms in my data we were partly moving towards the inquiry mathematics culture. But the old routines and habits were also stubbornly part of our interactions, and those were more compatible with the ways of talking in the school mathematics tradition.

5.4 Justifications must be based on the properties of mathematical objects

It was not common in the interactions of group A that we justified our mathematical claims (22 justifications in the peer interaction, and 8 when I was present). By a justification, I mean any kind of reason for one's claim. In group B the students were more active in this respect (76 justifications in the peer interaction, and 9 when I was present). However when I analyzed the nature of the justifications, I noticed that there was a sound ground for them. In the peer interaction of group A 18 of the 22 justifications were based on properties of mathematical objects. An example of that would be Anni's comment to Veikko when they were considering the relationship between the graphs of a function and its derivative function (problem 6).

Episode 6

77	Veikko:	Siis ... onko se nyt niin, että lähteekö ne niinkö samasta kohtaa niinkö nousuun [epäselvää]. Tai eihän ne nyt aivan samasta kohtaa, mut siis niinku, ö. Miten se nyt sitte on? (Anni naurahtaa). *Well ... is it so that, do they start increasing at the same x-value [indistinct]? Or not exactly at the same, but, I mean, in a way. How is it then? (Anni laughs a little bit).*
78	Veikko:	Tai siis jotenki silleen, että ne niinku vaan lähtee yhessä nou, tuolla. Ja ne laskee jotenki samaan pisteeseen ja. Niin. *Or something like that they just start increasing together there. And in a way they decrease to the same point and. Yes.*
79	Anni:	(pudistaa päätään) Mutta miten ne voi olla silleen, jos toinen on niinku kolmannen asteen ja toinen toisen asteen? *(shakes her head) But how could they do so, if one is third order and the other is second order?*
80	Veikko:	Miten ne sillon voi olla? *How can they be then?*
81	Anni:	Ni. *Yes.*

Veikko's suggestion that both of the functions increase and decrease together is disproved by Anni, who, in her justification, is most obviously referring to the different shape of the quadratic and third order polynomial functions.

In the group of boys, 50 of the 77 justifications were grounded on properties of mathematical objects, in 10 of the justifications a student was remembering a rule for doing calculations or manipulations of symbols, 10 of the justifications relied on the screen view of the calculator without any interpretation in words and in 5 of the justifications a student was referring to the authority of the teacher or textbook. Two justifications were mathematically irrelevant. An example of the last category is the claim that the formula $y = ax^2 + bx + c$ can not be a formula for constructing the equation of a parabola, because the formula $y - y_0 = k(x - x_0)^2$ is (an implicit supposition is that there is only one formula for that purpose).

All the justifications given by me were grounded on the properties of mathematical objects as well as all the justifications given by the students in my presence. In our justifications we were mostly producing the norm: *Explicit justification in mathematics must be based on the properties of mathematical objects.*

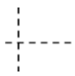
Yackel and Cobb (1996) describe the evolution of students understanding of what counts as an acceptable mathematical explanation and justification in their experimental classrooms in the early years of elementary school. Initially students' explanations may have a social rather than mathematical basis. As they participate in inquiry classrooms they come to appreciate mathematical reasons, and even more, they begin to differentiate between different types of them, for example those that describe procedures and those that describe actions on experientially real mathematical objects. Finally some students progress to making explanations as objects of reflection. The basis for these sociomathematical norms is the social norm that students explain and justify their thinking (Yackel and Cobb, 1996.)

In my data the situation seems to be different, perhaps because my students were experienced in high level mathematics. We were not accustomed to justifying our claims in small-group interactions. But when we did so, we were referring to properties of mathematical objects or otherwise justifying with logic and mathematical evidence (Lampert et al., 1996). I see that here the students are acting according to a traditional norm in our school and more generally in Finnish upper secondary education. It is expected that there is a mathematical basis for justifications. It only seems that the students think it is not their task to worry about justifying. And it seems that the teacher thinks it is not her task to justify claims in intimate teacher-students interactions. This suggests that, in order to develop inquiry cultures in my classrooms, it would be sufficient to start developing the social norms of challenging and justifying and at the same time supporting the existence of the already established sociomathematical norm of justification by the properties of mathematical objects.

The sociomathematical norm that explicit justifications should be based on the properties of mathematical objects can be seen to be part of the more general norm that mathematical discussion should be about mathematical reality, which was illustrated in section 5.3. Because the phenomenon was so clear in this case and explicit justifications are important in developing the inquiry mathematics tradition I devoted this section for the norm.

5.5 Written mathematical text should be exact and unambiguous

While working with the investigations I asked the students to write notes for themselves and, after each investigation, except the second one, I asked them to write a summary about their findings. I noticed the stylistic differences of, on the one had the speech of the students, and on the other hand the written text they produced.



Graph 4 The graph of the function $-\frac{x}{x+1}$

Episode 7

The boys were investigating what happens to the values of the functions f(x) = $-\frac{x}{(x+1)^2}$ (graph 3) and f(x) = (graph 4) when x approaches -1 (problem 2). Together they were discussing the second function.

89 Mika: No mitä siinä tapahtuu oikein? Lähestyy kumpaanki suuntaan äärettömyyttä. Vai?
Well, what happens there? Do they approach infinity from both directions? Or?

90 Juha: [epäselvää](näppäilee laskintaan)
[indistinct] (taps his calculator)

91 Reijo: Eikö tää ole toispuoleinen?
Isn't it uneven?

92 Mika: Niin. Niin, kato tästä se menee äärettömyyteen alas (liikuttaa kynäänsä pitkin kuvaajaa vasemmalta lähestyen -1:htä).
Yes. Yes, look, here it goes down to infinity (moves his pen along the graph from the left approaching x = -1).

93 Reijo: Niin se lähestyy, se lähestyy oikealta, niin se menee
Yes, it approaches, it approaches from the right, yes it goes

94 Mika: Ääretön niinku positiivisille ja sitte niinku vasemmalta ääretön niinku negatiiviselle.
Infinity for positive and then a kind of infinity for negative.

When speaking about the limit of the function the boys have the same graphs in their calculators. They have the same questions in their minds. And they have access to different modes of communication like gestures and so on. What is said in words may be difficult to understand by an outsider. But the boys write about the behavior of the function in a very different way.

118 Juha: Miten me kirijotetaan se.
How shall we write it down?

119 Reijo: Että jos se, jos, jos funktion, jos, jos, jos se x lähestyy Jos kun x
That if it, if, if the values, if, if, if the x approaches If, when x

120 Mika: No. Eikö funktion arvot, fu, eikö x:ä lähestyy, lähest, lähenee
Well. No, the values of the func, no, x approaches, appro,

121 Reijo: lähestyy lukua yksi, lukua miinus yksi oikealta puolelta, funktion arvot lähestyy
approaches number one, number minus one from the right side, the values of the function approach

122 Mika: Niin. Niin.
Yes. Yes.

123 Reijo: ääretöntä. Positiivista ääretöntä.
infinity. Positive infinity.

124 Mika: (kirjoittaa) Kun x:n arvot, [epäselvää], kun x lähestyy, lähestyy miinus ykköstä oi, vasemmalta eikö oikealta puolelta [epäselvää], niin funktion arvot lähestyy
(writes) When the x-values, [indistinct], when x approaches, approaches minus one from the ri, from the left, no, from the right [indistinct], then the values of the function approach

125 Juha: Lähestyvät. Mitä, positiivista ääretöntä?
Approach. What, positive infinity?

126 Mika ja Reijo: Ni. (kaikki kirjoittavat)
Yes. (all write)

127 Juha: x:n lähestyessä miinus yhtä vasemmalta puolelta
When x approaches minus one from the left side

128 Mika: Mie kirijotan, että vasemmalta puolelta menee negatiiviseen äärettömyyteen.
I am gonna write that from the left side it goes to negative infinity.

The boys write in their summary: When the x-values approach minus one from the right, the values of the function approach positive infinity. And when x approaches minus one from the left the values of the function go to negative infinity. In their written text the students struggled to use mathematical terms for the mathematical objects at hand in an exact way. Writing this way the boys were expressing and producing the norm: W*ritten mathematical text should be exact and unambiguous*.

Tatsis and Koleza (2008) found, in the communication of their pairs of pre-service teachers doing problem solving, the non-ambiguity norm: mathematical expressions are expected to be clear and unambiguous. In their data this norm was expressed through prompts for rephrasing. In my data a similar norm was produced in the context of writing mathematics, but not so much in the discussions of the students. I see this norm to be a traditional norm in our school. Teachers usually write notes on the board and show examples of tasks with clarifying comments and exact language. In the exams students are expected to write the same way. However, my students were not accustomed to speaking about mathematics, and perhaps thus the language they used was more informal.

5.6 A right answer ensures that the method is right

One of the investigations, the tangent and normal investigation (problem 5), was actually a situation of closed problem solving if we think of the answer, not of the method. The students were asked to find the equations of the tangent and the normal to the graph of the function $f(x) = \frac{1}{3}x^3 + x^2 + 1$ at $x = -1$. I had not yet taught the corresponding algorithms to the class. It was possible to check the answers by using the symbolic calculator. You can make the TI-92 calculator draw the tangent to a curve at a given x-value and you will also get the equation of the tangent. There is no direct function in the calculator for drawing a normal, but after you have its equation you can draw it in the same axis with the tangent and see whether they are perpendicular. Of course, the window of the calculator needs to be suitable.

Graph 5 The graph of the function f(x) = $\frac{1}{3}x^3 + x^2 + 1$ and the tangent and the normal drawn to the graph at x = -1

Episode 8

The students in group A have differentiated the function and evaluated the value of the derivative function at x = -1. Anni and Veikko interpret the result -1 to be the gradient of the tangent. Veikko takes his calculator and announces that he will start checking the answer. Anni sketches the graphs of the function and the tangent in her notebook and finds an approximate equation y = - x -1 for the tangent and she wants Veikko to check her answer. I come to visit the group. Veikko is explaining to Anni that the line x – 1 (Veikko may have misheard the expression of the line) has a wrong direction. I advise the students to draw a tangent to the curve at x = -1 with the calculator.

42 Opettaja: Entä jos sinä piirrät siihen kohtaan miinus yksi tangentin (Jenni ottaa laskimensa)? Voihan sillä laskimella piirtää sen ja tarkistaa, että
Teacher: What if you draw a tangent at x equals minus one (Jenni takes her calculator)? You can draw it with the calculator and check

And a little bit later I continue.

49 Opettaja: Siinä se on se tangentti, jonka yhtälöä etsitään. (Anni ja Jenni keskustelevat keskenään)
There you have the tangent, the equation which you are searching for. (Anni and Jenni talk to each other)

50 Veikko: Ko tossahan se näkyy se tommonen.
Coz, here you can see that.

51 Opettaja: Niin. Ja tuota niin, niin tavallaan sitähän voi käyttää apuna, mutta että teiän pitäs nyt laskien saaha se muodostettua. Mitäs nyt yhteyttä tuolla teiän laskulla on?
Yes. And, well, in a way you can use this as support, but you should find it manually. How is your solution connected to this?

In this extract I clearly suggest that the students should check their answer with the calculator. I express a belief that, at least in practice, a right answer is a strong indicator of an appropriate method and that a wrong answer shows the method to be wrong. The students find their answer to be wrong and they continue tackling the problem.

Jenni finds the formula $y - y_1 = k(x - x_1)$ from her notebook and the others agree with her that it can be used to construct the equation of a line. Together the students discuss the meaning of the symbols in the formula and use it for finding the equation of the tangent. They arrive at a result and compare it to the expression in the calculator.

171 Anni: Onkse tangentin yhtälö tollanen?
Is this the same as the equation of the tangent?

172 Anni: Hei! On! Eikö ollukki (katsoo Jenniin).
Look! It is, isn't it?(looks at Jenni)

173 Jenni: (nyökkää) Ym.
(nods) Yes.

174 Anni: Wautsi (naurahtaa)! Se on tuo yhtälö.
Wow (laughs a bit)! It is the equation

175 Veikko: No niinhän se onkin. Kato (näyttää Annille laskintaan, myös Jenni ottaa laskimensa).
So, well, it is! Look (shows his calculator to Anni, also Jenni takes her calculator).

176 Anni: Niin, niin.
Yes, yes.

177 Veikko: Kato, kato kato tuossahan tuo tangentti niin siinä oli
Look, look, look, here is the tangent, it has

178 Anni: No mie katon (naurahtaa).
Okay, I am looking (laughs a bit).

179 Veikko: siinä oli miinus x ja pilkku nolla
it has minus x and point zero

180 Anni: Jes (heilauttaa kättään)!
Yes. (waving her hand)!

181 Veikko: kuus, kuus, kuus, kuus seittemän. Hei me saatiin se, wiuuu (nostaa molempia käsiään kuin voiton merkiksi ja nauraa, molemmat tytöt nauravat).
six, six, six, six seven. Look, we got it! Wow (raises both his hands up like a sign of victory and laughs, both the girls laugh).

182 Veikko: Vitsi, [epäselvää] (riemuitsee vielä onnistumisesta, Anni nauraa).
Wow, [indistinct] (still rejoicing their success, Anni laughs).

183 Anni: Joo.
Yes.

184 Veikko: Hyvä Anni!
Well done, Anni!

185 Anni: Mikä on, Jenni sen kyllä vissiin keksi (osoittaa kynällään Jennia).
What? I guess it was Jenni who invented it (points with her pen at Jenni).

186 Veikko: No niin, Jenni ja Anni. Me oltas varmaan vielä eteen, eteenpäin viemässä [epäselvää].
So, well done, Jenni and Anni. Otherwise we would still work, be working [indistinct].

The joy of the students knew no boundaries when they discovered that their answer was right. They felt that they had succeeded in performing the task. They continued with the investigation and started considering how to obtain the equation of the normal.

Soon after this incident the students were constructing together that the normal to the curve at x = -1 passes through the same point as the tangent. Anni concluded that the gradient of the normal must be 1 because the gradient of the tangent is -1. The students use the formula $y - y_1 = k(x - x_1)$ to obtain the equation of the normal. They start checking their answer with the calculator. They draw the normal in the same axis with the graph of the function and the tangent.

100 Anni: (katsoo Veikkon laskimeen) Ei se oo se. Se ei mee kohtisuorassa. (Jenni ottaa laskimensa.) Niin ei tietenkään. Se mennee vaan silleen [epäselvää]. Eli se ei oo näin (pyyhkii vihkostaan laskut pois).
(looks at Veikko's calculator) No, it isn't that. It's not perpendicular. (Jenni takes her calculator) No, of course it's not. It just goes like that [indistinct]. So, it is not this (erases the working from her notebook).

Anni interprets the screen view of the calculator as if they have obtained a wrong answer. She starts erasing her notes, which I interpret to convey that she believes the method to be wrong. The students do not realize that a normal seems to be perpendicular to the tangent only when certain windows of the calculator are used. Jenni finds a formula $k_1 \cdot k_2 = -1$ which the students use to find the gradient of the normal. They obtain the same value 1. Anni is wondering:

154 Anni: No meillähän on, niin. Muttako jos siinon ykkönen, niin sehän jää silleen ykköseksi.
Well, we already have that. But if you have one, then it in a way stays one.

155 Veikko: Ym. (hiljaisuus, Veikko ja Anni tutkivat oppikirjaa) Nuo vaan sijoitetaan nuo pisteet sinne.
Yes. (silence, Veikko and Anni study their text book) We only substitute the points there.

156 Anni: Miksei siitä tuu oikea vastaus?
Why don't we get the right answer?

Only when I approve the students' answer, Anni starts believing.

186 Anni: Olikse muka se?
Is it really that?

I advise the students to check the answer by using a particular window of the calculator, a window which preserves the ratio 1 for the units on the axis. And this time, indeed, the normal and the tangent seem to be perpendicular to each other.

I interpret these events so that the students and I together are acting according to and producing the norm: *A right answer to a problem ensures that the method is right and a wrong answer means that the method is wrong.* This norm was also expressed and produced in the interactions of group B.

In traditional teaching in Finland students find the right answers for exercises in their mathematics textbooks. The students in my data, as well as I, myself, had many years' experience of the pattern of checking the rightness of a solution through the correctness of the answer. Furthermore, we upper secondary teachers often advise our students to use their graphics calculators for checking their answers in exams where they cannot find right answers other ways. But the norm produced through the pattern entails the focus on the answer instead of on the process. Pang (2001) analyzed differences in sociomathematical norms established in classrooms of two teachers, both employing student centered teaching methods. One difference was that in the lessons of the first teacher, excitement was expressed due to right answers, but in the other class producing only correct answers without mathematically justifiable process was rejected. Pang compares the two classrooms and concludes that the differences in sociomathematical norms in the two classrooms influenced the fact that in the first class opportunities for enhancing students' specifically mathematical ways of knowing were limited.

5.7 Summary of the sociomathematical norms

When investigating mathematics, one needs to approach questions in different ways than when just exercising tasks similar to what the teacher has just shown or that you can find in textbooks. Thus, new sociomathematical norms were negotiated or produced in the interactions of the two small groups:

- When investigating mathematics, one should approach the topic in a profound and creative way.
- When investigating mathematics, different approaches in addition to symbolic methods are approved.
- When investigating mathematics, questions about method, accuracy and writing down the solution are situation dependent.
- Mathematical discussion is about mathematical reality.

We also acted according to and produced old sociomathematical norms. Some of them were relevant in the new situation, such as:

- Explicit justification in mathematics must be based on the properties of mathematical objects.
- Written mathematical text is exact and unambiguous.

But some of the old norms were, however, not supporting the investigative approach:

- There are strict rules for a solution method and for writing down the solution and answer for a mathematical task.
- An accepted method for a mathematical task must be in a symbolic form.
- Mathematical discussion is about solving tasks.
- A right answer to a problem ensures that the method is right.

There are similarities in sociomathematical norms negotiated and produced in my data with norms reported in the literature, either in traditional classrooms or in inquiry classrooms. Two norms that I did not find correlates for, are about the need for a profound and creative approach and about the importance of following certain rules for method, accuracy and writing down the solution. The latter may reflect the context of Finnish upper secondary mathematics. In the final national examinations students are penalized for errors in accuracy, incomplete reporting of the solution method or using other than symbolic methods. I would think that the former is an important norm negotiated in the transition from school mathematics tradition, where solving tasks according to instructions is the main activity, to an inquiry mathematics tradition, where students are supposed to explore and create mathematics. It may be, however, that the activity of investigating mathematics in my experiment is a significantly different activity than creating solution methods for clear tasks in realistic mathematics education.

6. ACTING NORMS AND THE OCCURRENCE OF LEARNING OPPORTUNITIES

In this chapter I am going to answer the third research question: How were the participation structures and acting according to certain sociomathematical norms intertwined with the occurrence of learning opportunities. On the basis of my analysis I shall present the major or most interesting features of those concrete interactions that contributed to the occurrence of learning opportunities and the ways they did this.

6.1 Participation structures and learning opportunities

6.1.1 Peer interaction

Working together

In group A the participating students normally worked in a collaborative way and strived for consensual conclusions. Anni and Veikko expressed their thinking. Jenni was following the discussions. Usually the students were listening to each other and showing agreement. These characteristics of their interaction brought forth a lot of learning opportunities for all the students.

Episode 1

The group is considering what happens to the values of the function $\frac{\sin x}{x}$ when x approaches zero (problem 1). In their prior discussions the students had built a common understanding of the mathematical situation. Just before the episode, they had been discussing the fact that in the decimal expressions of the values of the function there are more and more digits nine when x gets closer and closer to zero.

347 Veikko: Pitikö meiän täältä
Do we need to

348 Anni: Ne vaan koko ajan, tuota tullee lähemmäks sitä. Ei. Nehän mennee lähemmän ykköstä. Niin.
All the time, well, they get closer to it. No. They get closer to one. Yes.

349 Jenni: Ym (nyökkää).
Um (nods).

350 Veikko: Niin.
Yes.

351 Anni:	Niin. Että tää ei, pääseekö tää koskaan ykköseen? *Yes. So that it does not. Does it never get to one?*
352 Veikko:	Ei. *No.*
353 Anni:	Niin. Tää ei pääse ykköseen (Anni sanoo ”niin” niin hämmästyneellä äänellä, että häntä alkaa naurattaa ja kaikki muutkin naurahtavat). *Yes. It never gets to one (Anni's ”yes” was said in such astonishment that she begins laughing and all the others join her).*
354 Veikko:	Se ei pääse ikinä siihen ku se ei mee nollaan, ku se hyppää sen yli. *It never gets there coz it doesn't go to zero, coz it skips it.*
355 Anni:	Aivan. *Exactly.*
356 Anni:	Laitetaan, että … Se aina vaan lähe (naurahtaa). *Let's put it, that ... It all the time appr (gives a short laugh).*

I interpret the discussion so that Anni starts thinking about the function values as numbers, instead of focusing on the digits. She sees that the values of the function get closer and closer to one (348). In her enthusiasm, she interrupts Veikko from finishing his sentence. But Jenni and Veikko listen to Anni and show that they understand what she means and that they agree with her (349, 350). Anni elaborates her conjecture and Veikko is supporting her. Then Veikko offers his own interpretation (354), which is accepted by Anni (355). The students had been and still were working to find a taken-as-shared interpretation of the phenomenon and a common conclusion. A learning opportunity occurred for all of them to understand that the values of the function approach the number 1.

Also in group B a great number of learning opportunities occurred to the students when they expressed their own thinking, were listening to others, were showing agreement with each other and collaborating. One specific feature in the interactions of this group, which contributed to the establishment of collaboration and thus caused many learning opportunities, was the democratic leadership of Mika. He was a mathematical authority in the group, at least Reijo and Pekka gave him this status. Very often Mika lead the group, in resonance with the other members, to work with the tasks and to discuss the mathematical themes. He helped the other boys to follow and answered their questions. And he didn't take the competition with Juha too seriously. There were moments when Mika's leadership was not democratic, but most of the time he behaved in a very constructive way to promote the mathematical discussions in the group and the learning of all the members of the group.

Episode 2

The boys are constructing a method for finding the equation of a tangent to a curve at x = -1 (problem 5). They have evaluated the gradient of the tangent and found the formula $y - y_0 = k(x - x_0)$ from their notebooks. Together the students had discussed that, in addition to the gradient, they need one point from the graph. In order to get the point they should substitute x by -1 in the expression of the original function. Juha and Pekka did not bother to do the substitution manually. They were waiting for Mika and Reijo to do the work.

125 Mika:	Nyt meillon se piste siinä (opettaja poistuu). *Now we have the point (the teacher leaves the room).*
126 Pekka:	Ym. *Um.*
127 Mika:	Niin. *Yes.*
128 Reijo:	[epäselvää] viimeks pois sielä juhlasalissa? *[indistinct] last time from the gym?*
129 Pekka:	Oli. *Yes.*
130 Reijo:	Ai (naureskelee). *Oh (gives a short laugh).*
131 Juha:	Mika, what´s that (osoittaa Mikan vihkoon)? *(in English) Mika, what's that (points to Mika's notebook)?*
132 Mika:	No se, varmaanki se y-koordinaatti. Ko joka äsken laskettiin. *Well, it must be the y-coordinate, which we just calculated.*
133 Reijo:	Ai onkse y_o se y-koordinaatti, mikä me äsken laskettiin? *Oh, is y_o the y-coordinate, which we just calculated?*
134 Mika:	On. *Yes it is.*
135 Reijo:	Ym. Ja se x_o se x-koordinaatti, mikä me laskettiin, se x. Ym. (ryhtyy laskemaan) *Um. And x_o is the x-coordinate, which we calculated, the x. Um. (starts calculating)*
136 Mika:	Se miinus ykönen. *That minus one.*
137 Mika:	Kyllä nyt, jos noita muistas vielä, että miten niitä laskettiin *Well, now, if you only remember how we did these.*

At the end of the episode (131 – 137) Mika is readlily helping Juha and Reijo in understanding the meaning of the result of their previous calculations. This is the case even though just a few moments previously he was irritated about the behavior of Juha and Reijo when they were quarreling. Juha and Reijo had a learning opportunity to understand the meaning of the symbols x_0 and y_0 in the formula.

All the students from the two groups, who returned their diaries (Pekka did not), recognized that their learning was supported by the investigative small-group approach. Collaboration was an inseparable part of the method and there were not too many severe problems in either group with this respect. After the experimental course Jenni wrote in her diary:

> "Mielestäni tämän kurssin työskentelytavat ovat olleet todella hyviä. Kun on saanut ensin ryhmässä pohtia ongelmia, asiat on tajunnut paljon helpommin ja on ollut hauskaakin."
> *"I think the working methods in this course were really good. After we had the opportunity of thinking through the problems together with the other group members, I understood the topics better and we even had fun."*

Mika, from the other group, made a similar comment:

> "Opin paljon, ryhmätöistä etenkin. Tutkiminen on hyvä oppimistapa."
> *"I learnt much, especially during the group work. Investigating is a good way of studying."*

The detailed analysis, however, shows that in both of the groups many learning opportunities were destroyed because someone was not listening to what someone else said. In group B the reason was most often that one or two boys, very often Juha among them, were acting alone. The person or persons did not collaborate or adjust their actions according to the others. In group A Jenni's retirement often led her to work alone. Acting individually hindered the students from gaining learning opportunities that occurred in the group for other students.

I have not found many research reports which examine the relationships between social and sociomathematical norms and learning, or learning opportunities. Herbel-Eisenmann et al. (2003) evaluate the research done in the emergent perspective. They conclude that most of the articles and book chapters published in this tradition work in the individual cells of the table in figure 1 in section 2.4.2, but do not look more specifically across the rows. I see that in many research reports it is supposed that the social and sociomathematical norms compatible with an inquiry mathematics tradition promote conceptual learning in mathematics. But more specific empirical investigations of these relationships are rare.

Tatzis and Koleza (2008) studied social and sociomathematical norms in discussions of pairs of pre-service teachers during problem solving, and what the effects of norms were on the solving processes. They argue that norms influence how a mathematical concept or method is established. According to them the collaboration norm plays the most vital role in the thematic development of the discussions, since it is involved in all stages of it. On the other hand, acting alone means breaching the collaboration norm and destroying the collective process (Tazis and Koleza, 2008).

Cobb (1995) explains the change in his and his colleagues' views when working with European researchers. Originally they had been, in a Piagetian way, focused on the learning opportunities that arise as children in small-groups try to resolve conflicts in their individual view points. But later they modified their rationale for small-group work to take account of the learning opportunities that arise for children "as they mutually adapt to each other's activity and attempt to establish a consensual domain for mathematical activity" (p. 26). He thus sees collaboration, the attempt to establish consensus, as a basic presupposition for the opportunity of learning in small groups.

According to Webb (1989) help-giving which involves explanation, not just information, enhances the understanding of the help-giver, who is most often an able student. Whether help is beneficial to the target student depends on its relevance, level and timing and whether the student understands the explanation and has a chance of using the explanation to solve the problem. If the roles of "helper" and "helped" are played consistently, there is a danger of a "caste" system which discourages the helped from active participation (Mulryan's unpublished dissertation sited in Good et al., 1992).

Justifying, disagreeing and asking questions

In group A there would most probably have occurred more and deeper learning opportunities if the students had justified and challenged their claims more often. At the end of the first derivative investigation (problem 4), the students had evaluated average velocities for smaller and smaller time intervals.

Episode 3

99	Anni:	Hei nyt mie alan nähä tässä jotain. *Look, I am beginning to see something here.*
100	Veikko:	Tässähän tullee nollia vaan lissää. *There are more and more zeros.*
101	Anni:	Niin. *Yes.*
102	Anni:	Tuohon varmaan tullee ykkönen muttako tuossa ei näy ne kaikki [epäselvää]. *There must be one here, but I can't see them all [indistinct].*
103	Veikko:	[epäselvää] niitten keskiarvo [epäselvää] 0,046 metriä sekunnissa. *[indistinct] their average [indistinct] 0.046 meters per second.*
104	Anni:	Ym. *Um.*

105 Jenni: Siis pitäskö siihen tulla 0,0462 metriä per sekunti (Veikko vilkaisee Jenniin)?
Should it be 0.0462 meters per second (Veikko looks at Jenni)?

106 Veikko: Eli tästä voidaan päätellä, että se kulkee 4,6 cm sekunnissa (katsoo Jenniin). Eiku, niin. Eikö vain?
So, we can draw the conclusion that it goes 4,6 cm per second (looks at Jenni). No, yes. Can't we?

107 Anni: Ei vaan se hetkellinen nopeus ...
No, but the instantaneous velocity ...

108 Veikko: On 46 eiku 4,6
Is 46, no 4.6

109 Anni: yhen sekunnin kohdalla on 0,046.
at one second it is 0.046

110 Veikko: Kaks pistä siihen vielä.
Add still two.

111 Anni: Mitä että?
What?

112 Veikko: No eikö ne oo metriä sekunnissa?
Isn't it meters per second?

113 Anni: Vaan onko tää se
Or is it the

114 Veikko: senttimetriä, 46 millimetriä (osoittaa kynällään Annin vihkoon).
centimeters, 46 millimeters (points with his pen to Anni's notebook).

115 Anni: Onkohan tää nyt ihan okein (naurahtaa)?
I just wonder whether it is quite right?

116 Veikko: Siis jos se y-akseli oli kerta metriä. Matka metrejä. Eikö vaan? Eikö ollukki ne metrejä?
So if the y-axis is in meters, distance in meters. Right? Isn't it in meters?

117 Anni: On.
Yes.

118 Veikko: Nyt sitte tiietään arvonki.
Now we know also the value.

119 Anni: No ni. Sitte vielä tää viimenen (osoittaa monisteeseen).
Okay. Then this last one (points to the worksheet).

I interpret that Jenni saw the average velocities approach the limit 0.0462 (the exact value), which she thinks to be the instantaneous velocity at one second (105). This view is supported by the observation that in the next episode she is the only person to answer, after a moment of silence, when I ask what happens to the average velocities when the time interval is made shorter and shorter. She says: “Siis koko ajan lähenee tuota 0,0642:sta”. (“It is all the time approaching that 0.0642.”) The word “lähestyä” (”approach”) was used by the students to refer to the limit process. Perhaps Anni also saw (99) something similar, but Veikko did not let her explain it and Anni did not return to her idea. On the contrary, Veikko and Anni use the number 0.046 for the instantaneous velocity. In turn 103, Veikko hints that he is taking some kind of average of the average velocities. I interpret that Veikko is thinking that average velocities, when taken in a short enough interval, are good approximations for the instantaneous velocity. Nevertheless, the next episode suggests that neither Veikko nor Anni had been thinking of average velocities approaching a limit. If they had, one of them would more likely have answered me instead of Jenni.

The students don’t see their differing views. They just discuss the answers, but they don’t describe how they got the answers and they don’t justify their methods. Had they done so, they would have faced the interpretations of others and they would have had to challenge either their own or others’ thinking. There are obviously ingredients for interesting discussions in this episode, but the students did not go deep enough. Now there occurred a learning opportunity only for Jenni who constructed the important idea. Anni and Veikko had no learning opportunities. Silencing Jenni may also contribute to the lack of argumentation. But here Veikko, however, looks at Jenni after her comment. So, with particular intensity he has noticed her utterance. This interpretation is also supported by the utterance 110.

In group B the students used several different strategies for communication when investigating mathematics. The students justified their claims more often, they disagreed with each other more frequently and there were fewer utterances of agreement than in group A. The students in group B also asked more questions, when they did not understand the thinking of the others or what was going on in the group. These features of the interaction were included to a great extent in situations where learning opportunities occurred to the students.

Episode 4

The students in group B were investigating what happens to the values of the function $\frac{\sin x}{x}$ when x approaches zero from the right and the left (problem 1). They had constructed the required sequences of x-values. From the graph of the function the boys had already concluded that the values of the function approach one. I had advised them to make some numerical calculations to show the case.

219 Pekka:	Otetaan vaikka näin: kolome ja kaks ja puoli, ja kaksi, yks ja puoli, ja *Let’s put it like this: three and two and a half, and two, one and a half, and*

220 Mika: No ei kai nytte vähän pienempiä lukuja vois ottaa tuota
But no, I think you can put smaller numbers than that, well

221 Reijo: Miinus. Pitäs [epäselvää]
Minus. You should [indistinct]

222 Pekka: Niin, mutta jos se oikealta puolelta tullee (osoittaa kynällään kuvaajaa laskimessaan).
Yes, but if it comes from the right side (points with his pen to the graph in the calculator).

223 Reijo: Meni vasemmaltaki.
But it came from the left side as well.

224 Pekka: No niin, mut se, tekkee taas siitä.
Oh no, but it, it does it again.

225 Mika: Niin, mut se [epäselvää] yhtä aikaa pistää siihen.
Yes, but [indistinct] put it at the same time there.

226 Pekka: Yks ja puol, yks
One and a half, one

227 Reijo: Katoppa miinus kolme. (kirjoittaa)
Look at here, minus three. (writes)

228 Mika: Kai se voi olla vähän pienempiä nää luvut, että sä huomaat, että ne lähestyy ykköstä (osoittaa kynällään Pekkan vihkoon).
The numbers could be a little bit smaller, so that you see them approaching one (points with his pen to Pekka'a notebook).

229 Pekka: No kai mie sen voin olla mut. Pienempiä, on mulla tarpeeksi, puoli.
Well, I guess I can make. Smaller? I have small enough, half.

230 Mika: Puoli, ei taia olla ihan pieni ku se lähestyy niinkö siinä tuhannesosisa. Niin.
Half is not small enough, because it approaches in thousandth parts. So.

231 Pekka: Pistä sie siihen, että yks pilkku nolla, nolla, nolla, nolla, nolla, nolla, nolla, nolla, nolla, nolla yks (Reijo naurahtaa).
Why don't you put one point zero, zero, zero, zero, zero, zero, zero, zero, zero, zero, one? (Reijo gives a short laugh)

232 Mika: No niin pistänki.
Well, I'll put it like that.

233 Reijo: Hm.
Um.

234 Mika: Mä laitan ykösestä.
I'll make it from one.

235 Pekka: Kolomosesta sama.
From three, all the same.

236 Mika: Tai kakosesta.
Or from two.

237 Pekka: Kolomosesta. (pojat laskevat funktion arvoja)
From three. (the boys evaluate function values

Pekka starts by constructing x-values which are terms of an arithmetic sequence (219 – 229), and which finally end up at one half. Mika challenges his sequence (220). Pekka justifies his solution (222) by saying that his numbers come from the right. Mika suggests that the x-values should be smaller so that one can see that the function values approach one (228). Pekka defends his sequence and justifies it by saying that he has small enough numbers, he has one half (229). Mika challenges this idea by claiming that half is not small enough. I interpret that, in turn 230, Mika refers to geometrical approaching, as in the sequence 1/10, 1/100, 1/1000, …, which they had constructed at the beginning of the session. Pekka gets provoked by Mika and attacks him (231). But his comment shows understanding of Mika's point. I interpret that at least Pekka, perhaps Reijo too, had a learning opportunity to deepen their conception of numbers approaching a limit.

Challenging others' ideas by disagreeing and justifying one's own ideas make explicit the thinking of many different persons. On those occasions interactions can be said to be multivocal (Cobb, 1995). In his research on second grade children working in small-groups Cobb found that multivocal interactions were usually productive in that learning opportunities arose for children.

Competition and disputes

The competition between Mika and Juha, the disputes and the boasting in group B, destroyed some learning opportunities. When the interaction changed from genuine mathematical discussion to a particular kind of social game of arguing and boasting, the goal was no longer trying to understand the mathematical theme or thinking of others. The most important thing in those situations was gaining status by showing one's superiority and by winning the opponent. This prohibited genuine mathematical discussion and destroyed learning opportunities.

Episode 5

The boys have been working with the last investigation that deals with the connections of a function and its derivative function (problem 6). In our discussions the boys have concluded that when the derivative function is negative the original function is a decreasing function and vice versa. While discussing with the boys, I had told them that the values of the derivative function are the gradients of the tangents to the original function. After a few minutes delay, the same idea strikes Mika.

33 Mika: (osoittaa kynällään derivaattafunktion kuvaajaa) Tiiättäkö muuten, tässä on m, saattaa nähä että palijoko se kulmakerroin on!
(points with his pen to the graph of the derivative function) By the way, do you know, here we, you can see how big the gradient is.

34 Reijo: (aukaisee suunsa suurelle) Miten? (Juha heittelee kuivaa sientä)
(opens his mouth wide) How? (Juha is playing with a dry sponge)

35 Mika: No tuosson, että, jos tuosson kaks (osoittaa pistettä kysymyspaperin derivaattafunktion kuvaajalta), niin siinä kohtaa se kulmakerroin on kaks (näyttää kohtaa laskimen kuvaajalta, katsoo Juhaa, joka kohottelee kulmakarvojaan). Se taitaa olla niin. Vähä mie oon guru.
Well this is, that, if this here is two, (points to a point on the graph of the derivative function on the worksheet), then at that x-value the gradient is two (shows a point on the graph in the calculator, looks at Juha, who raises his eyebrows). It may be like that. What a guru I am!

36 Juha: Siis tuota mi. No nyt sitä ei kuitenkaan tarvitte selevitä, guru.
Well, what? But we don't need to clear it up now, guru.

37 Reijo: No ni, no niin sano jo.
Now, tell us.

38 Juha: Mie voin tulla viikonlopuks väittelemään teille asiasta.
I can come to your place for the weekend to dispute it.

39 Mika: Väitelläänpä loppuun se
Let's finish the argument

40 Juha: Väitellään, väitellään, saateri.
Let's argue, let's argue damn.

41 Mika: Jätkä, miten väittää, että miinus yks on yhtäkuin yks.
Man, how can you argue that minus one equals one?

42 Reijo: Joo, joo, ei oo, ym, ym.
Okay, okay, it's not. Um, um.

43 Juha: Ei, ei, ei, ei. Älä, älä, älä kuule.
No, no, no. Do not, do not

44 Juha: Otetaan sitte tuo kamera (nyökkää kameraan päin) mukaan. Filimataan se (naurahtaa). Opettaja saa materiaalia, miten kaks paukapäätä väittelee matematiikasta.
Let's take the camera with us (nods to the camera). Let's film it (gives a short laugh). The teacher gets material how two meatheads argue about mathematics.

45 Mika: Kaks paukapäätä? Yks paukapää vastaan guru (Juha ja Reijo nauravat).
Two meatheads? One meathead and a guru. (Juha and Reijo laugh)

46 Juha: No niin, jätetäänpä nyt kliseet pois. Tehhään tuota (nyökkää Reijoun päin).
Well, let's leave all the clichés. Let's work with that (nods to Reijo).

Mika gets a brilliant idea, and remarks on its value for himself, too (33 – 35). His mistake is to show his enthusiasm and to utter the last sentence: "I must be a guru." This triggers the competition and dispute mode in the group. Juha refuses to consider the idea seriously (36 – 38). The discussion is led to an old dispute of the boys and Mika's great mathematical idea is lost. The learning opportunity to see what Mika saw was hindered from Reijo and Juha.

Sfard and Kieran (2001) show, how, in a pair of 13 year old boys working together in a reform classroom, the boys had different focuses. The other boy was mainly preoccupied with object-level issues, i.e. solving the mathematical problem, and the other was more concerned with his positioning in the discourse and thus mainly interested in the interaction itself. These preoccupations had an impact on how the boys managed their private channels of interaction and how well they functioned on the object-level.

In the case of the previous episode, the focus of the discussion shifted from object-level to the interactional level. Finally the boys no longer functioned at the object-level at all, and thus a learning opportunity was lost.

The assertiveness of Veikko and retirement of Anni and Jenni

Veikko, the male member in group A, expressed his own thinking very eagerly. Often he had difficulties in listening to the girls and sometimes he took the floor from them and even interrupted them. He was the student who mostly represented the group in the discussions with me. This assertiveness of Veikko had an influence on the fruitfulness of the collaboration. Some important learning opportunities in the peer interaction of the group were hindered when Veikko interrupted Anni or started talking about his own thinking just after Anni had expressed a promising idea. A striking example is taken from the first derivative investigation (problem 4).

Episode 6

The students were considering the meaning of the gradient of the chord (f(z) – f(1))/(z – 1) for a time-distance graph.

31 Anni: Siis mitä nuo ny meinaa? (vilkaisee vihkon edellistä aukeamaa)
Ko tää on aika ja tuo on matka (osoittaa laskimensa koordinaatiston akseleita vuoron perään).
So, what do they mean? (looks at the previous two pages of her notebook) Because this is time and that's distance (points to the axis in her calculator).

32 Jenni: Niin miten se piirrettiinkään? (ottaa oman laskimensa)
So, how do we draw it? (takes her calculator)

33 Veikko: Niin mitä tuo sekantin kulmakerroin tarkoittikaan sitte? (katsoo hieman voitonriemuisena Annia) Siis ajan ja [epäselvää].
What does the gradient of the chord mean, then? (looks at Anni triumphantly) Because it is time [indistinct].

34 Anni: (ei huomaa Veikkon ilmettä) Onko se silleen niinku keskimääränen joku tommonen. ... En minä tiiä.
(does not notice the expression on Veikko's face) Is it something like an average, something like that. ... I don't know.

35 Anni: (Jenni seuraa Annin ja Veikkon keskustelua)
Mut eikö
(Jenni is following the discussion between Anni and Veikko)
But, isn't it,

36 Veikko: Ko aika kasvaa
When time goes on

37 Anni: eikö sitä tuu keskimääräinen
isn't it average

38 Veikko: Kato nyt ko tämä on niinku aika (osoittaa vihkoaan). Niin, niin, siis x on, niin, niin se kohtaa sitte täällä sen (osoittaa Annin laskimen näytössä olevaa kuvaa).
Se niinkö [epäselvää] keskimääräinen arvo tai silleesti.
Look, here, because this is time (points to his notebook). Then, then, well x is, yes, here it meets that (points to the graph in Anni's calculator). The [indistinct] average value or something like that.

39 Anni: Ym.
Um.

40 Veikko: Siis miten se nyt meni?
Well, how was it, then?

41 Anni: (osoittaa kynällään laskimen näyttöä.) Niin, että jos se on se keskimääräinen arvo, niin sitte että mitä jyrkempi se on, niin sitä pitemmältä ajalta se on otettu. Se keskimääräinen homma.
(points with her pen to the screen of the calculator). So that if it is the average value, then the steeper it is, the longer is the time. For the average thing.

It seems to me that Anni was close to constructing an important idea in the investigation (31 – 37), that the gradient of the chord is the average velocity. But Veikko interrupts her and, by doing this, transforms the meaning of what Anni was saying (38). Anni gives up and returns to the previously discussed idea, that the longer the time interval is, the steeper the corresponding chord (41). Most probably, a learning opportunity for all the students was destroyed. A few times it happened that Anni was expressing a promising idea and Veikko prohibited it from being heard so that a learning opportunity was lost. Usually Anni did not persist with her idea, like the boys in the other group sometimes did.

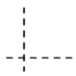
The assertiveness of Veikko also contributed to the silencing of Jenni. Sometimes even Anni took part in it. This led Jenni to work alone with her calculator or with her notes. And by not following the discussions of the others in the group, she lost learning opportunities.

Veikko, in group A, used his social authority (Cobb, 1995) to influence the way the students in the group discussed mathematics. He was not mean and the atmosphere in the group was very friendly most of the time. Tannen (1993b) argues that if one speaker in a conversation repeatedly overlaps and another repeatedly gives way, then the resulting communication is asymmetrical and the effect (not necessarily the intent) is domination. Here the students seem to produce a typical power imbalance observed in cross-sex conversations by, among others, West and Zimmerman (1983). It is a loss for the collective activity, if some participants are deprived of their possibility of contributing in small-group discussions. In group A, the loss was even worse because Anni seemingly had a capacity for producing sophisticated mathematical ideas, and Jenni too. Some of their potential was prohibited from being realized in the collective activity and thus learning opportunities were lost.

Roles of the students in knowing and learning

The investigative approach in studying mathematics requires that students do their own observations and conclusions, and construct by themselves mathematical meanings connected to the phenomena at hand. This is contrary to the expectations placed on students in their previous studies of mathematics. Traditionally, the role of students in Finnish schools has been to follow and try to understand the presentation of the teacher and apply the mathematics taught, normally by solving tasks. At first, the exercises given to the students are very similar to the examples the teacher has just shown to them; later applying the methods and principles in new situations may be required.

In both of the groups traditional roles of students in terms of knowing and learning caused problems for acting in a relevant way in the new situation, investigating mathematics in small-groups. This was especially in the beginning of the course. Acting according to the old norms hindered learning opportunities from the students.



Graph 6 The graph of the function f(x) = $\frac{1}{x}$

Episode 7

The students in group A were asked to find out what happens to the values of the function f(x) = when x increases above any boundaries or decreases below any boundaries (problem 3). At the beginning of the episode the students have not yet drawn the graph of the function. They start their discussion by focusing on the symbolic expression of the function. The graph of the function is shown above (graph 6).

15 Anni: Sillonku tuo x kasvaa, niin se funktio pienenee.
When that x increases, the function decreases.

16 Veikko: Niin.
Yes.

17 Anni: Niin ja sitteku x pienenee, niin se kasvaa.
Yes, and when x decreases, then it increases.

18 Veikko: Niin mutta sittekö se alkaa, pienenee negatiivisessa suunnassa. Eikse taas kasva sillon? [epäselvää] (ottaa laskimen)
Yes, but when it begins, it decreases in the negative direction. Doesn't it increase again? [indistinct] (takes his calculator)

19 Anni: Eikö se, pienene? Vai? (Veikkon ilme sanoo: En tiedä)
Doesn't it decrease? Or? (Veikko's expression says: I don't know)

20 Anni: Sillonku x kasvaa (kirjoittaa, samoin Jenni, ja Veikkokin ryhtyy), niin funktio pienenee. Mutta mitä jos se kasvaa sinne negatiiviseen, pieneneekö se. No mut sillon se suurenee.
When x increases (writes, also Jenni and Veikko start writing), then the function decreases. But what if it increases to the negative, does it decrease then? But then it increases.

21 Veikko: Mut sittekö se on tuon yli yhen kohalla ja siitä
But when it is at one and to

22 Anni: [epäselvää]Jos se on vaikka miinus kaheksan, sitte se ois miinus
[indistinct] If it's minus eight for example, then it is minus

23 Veikko: Eikö se on niinku tommonen (näyttää laskimessaan olevaa kuvaajaa tytöille).
Isn't it something like this (shows the graph in his calculator to the girls).

24 Anni: Jos se kasvaa, niin se on miinus (katsoo Veikkon laskimen näyttöä). Niin.
If it increases, then it is minus (looks at the screen of Veikko's calculator). Yes.

25 Veikko: Eli sillonku se lähestyy nollaa, niin se kasvaa rajattomasti.
So, when it approaches zero, then it increases boundlessly.

26 Anni: Niin se kas, niin. (pyyhkii pois jotain vihkostaan)
Yes, it in, yes (erases something from her notebook).

27 Veikko: Siis lähestyy nollaa.
So, it approaches zero.

28 Anni: (Veikkolle) Kummalta suunnalta?
(to Veikko) From which direction?

29 Veikko: Ihan kummalta, no niin, joo. (katsoo kuvaajaa) Sillonku se lähestyy silleen niinkö, otetaan x miinus ykkönen, niin sillon se kasvaa miinus äärettömään.
From either, okay, so yes (looks at the graph). When it approaches like, let's take x-value minus one, then it increases up to minus

30 Anni: [epäselvää]
[indistinct]

31 Veikko: asti. Sillonku sitä lähestytään oikealta päin, niin sit se menee sinne äärettömään. … Onko se niinkö miinus, x miinus? Onko se x miinus, et se tulee niinkö sieltä toiselta suunnalta? Voiko sen laittaa siihen x:ään, vai mihin se pitää laittaa? (Jenni katsoo Veikkoa) Eikö siinä on se x, ko x lähestyy. Okei, no niin, ei mitään. (Anni naurahtaa)
infinity. If you approach it from the right, then it goes to infinity. ... Is it minus, x minus? Is it x minus, so that it comes from the other direction? Can you put it to the x, or where should you put it? (Jenni looks at Veikko) I mean there is the x, because x approaches. Okay, well, never mind. (Anni gives a short laugh)

32 Anni: Sillonku se x lähestyy nollaa negatiiviselta puolelta
When x approaches zero from the negative direction

33 Veikko: Niin.
Yes.

34 Veikko: Negatiiviselta puolelta, niin mitä sille funktiolle tapahtuu? Niin raja-arvo, tai ei oo raja-arvoa. Siis niinkö
From the negative direction, what happens to the function? Well, the limit, or, there is no limit. I mean

35 Anni: Se funktio pienenee.
The function decreases.

36 Veikko: Pienenee rajattomasti. (kirjoittavat)
Decreases boundlessly (the students write).

37 Anni: Ym.
Um.

38 Veikko: Miinus äärettömään. Ja kun se lähestyy oikealta, sit se kasvaa rajattomasti. (kirjoittavat)
To minus infinity. And when it approaches from the right, then it increases boundlessly (all write).

In the beginning, Anni constructs and Veikko agrees that when x increases, the values of the function decrease (15 – 16). Thinking only about positive numbers, Anni states that when x decreases, the values of the function increase. The students are unsure about what happens to the values of the function when x decreases "in the negative direction" (18 – 24). In turn 25 the discussion switches to deal with the question of what happens to the values of the function when x approaches zero. When x approaches zero from the right the values of the function go to infinity and when x approaches zero from the left the values of the function go to minus infinity. During the previous lessons we had been practicing tasks about determining limits of infinity and negative infinity at certain x-values with the help of graphs. The students find a conclusion accordingly (31 – 38). They did not explore mathematics new to them but repeated a method just taught to them. Now, when studying in an investigative way, acting like this destroyed a learning opportunity to construct new ideas about limits at infinity and minus infinity.

In both of the groups, when the students tried to apply knowledge in the textbook or remember principles from physics instead of investigating, this destroyed learning opportunities for the students. They did not understand the text or they applied physics in irrelevant ways.

Episode 8

The students in group B have concluded that on a time-distance graph the average velocity is the gradient of the chord. Juha has an idea for finding the instantaneous velocity.

18	Juha:	Jos me otetaan se i-secti, niin haetaan se sekantti, mikä koskettaa sitä kuvaajaa yhesä kohasa elikkä niinku [epäselvää]. *What if we take the i-sect, I mean if we find the chord which touches the graph at one point. So [indistinct].*

Juha's idea, which was based on their previous working, was not accepted by the group. But later Pekka adopts it and elaborates it.

107 Pekka:	Eikö se ollu joku sellanen, että (ottaa kynän käteensä) *Wasn't it something like, that (takes a pen in his hand).*
108 Reijo:	Tangenttisysteemi. *Tangent system.*
109 Pekka:	(piirtää samalla) Niinkö, jos tässon vaikka tangentti näin. Ja tässon nämä systeemit (piirtää käyrän). *(draws while talking) Well, if we have a tangent, this way, here. And here are these systems (draws a graph).*
110 Mika:	Eikö se ollu näin *Wasn't it so that*
111 Pekka:	Eikö se ollu joku are, alue tässä näin (näyttää kynällään aluetta piirroksessaan). *Wasn't it some are, area, here like this (shows with his pen an area in the drawing).*
112 Mika:	Niin. (Reijo nyökkäilee) *Yes. (Reijo is nodding)*
113 Pekka:	Se oli sen pinta-ala. *It was its area.*
114 Mika:	Se oli pinta-ala. (Juha katsoo Pekkaa) *It was area. (Juha looks at Pekka)*
115 Pekka:	Niin. *Yes.*
116 Mika:	Niin. *Yes.*
117 Pekka:	Elikkä meiän pitäs laskea niinkö. Miten me saatas se, sieltä tulee semmonen. Miten saa tuon pinta-alan sitte laskettua? *So we should find out. How could we get the, it becomes. How can we calculate the area, then?*

The boys find the area under the chord to the graph in the time interval [0, 1] seconds. They decide that their answer is the instantaneous velocity at one second. When I come and discuss with them they justify their method.

304 Pekka: Niin. Ja se kerrotaan tuolla ja sitte jaetaan kahella. Kolmion pinta-ala.
Yes. And then it's multiplied by that and divided by two. The area of a triangle.

305 Mika: Niin.
Yes.

306 Opettaja: (vähän hämmästyneenä) Kolmion pinta-ala?
Teacher: *(looks astonished) The area of a triangle?*

307 Mika ja Pekka: No niin, eikö se oo.
Well, yes. Isn't it?

308 Mika: Semmosta on fysiikassa, ykösessä opetettu (Reijo naurahtaa).
We have been taught so in physics, in the first course (Reijo laughs a bit).

309 Juha: Kes, keskinopeus niinku [epäselvää].
Aver, like average speed [indistinct].

310 Pekka: Ja keskinopeus oli sitte tuo.
And then average speed was this.

311 Opettaja: Keskinopeus oli mikä?
What was the average speed?

312 Pekka: Se on se sekantti.
It is the chord.

313 Opettaja: Sekantti ja sen kulmakerroin.
The chord and its gradient.

314 Pekka ja Mika: Ym.
Um.

315 Juha: Niin, mut jotenki mullon semmonen
Yes, but I have got a feeling

316 Opettaja: Ja hetkellinen nopeus on sitte pinta-ala (vähän kysyvästi).
And instantaneous velocity is then area (questioning expression on her face).

317 Pekka: No niin mä ainaki, me ainaki muisteltiin.
Well, at least I, at least we remember.

318 Mika: [epäselvää] (Reijo naurahtaa).
[indistinct] (Reijo gives a short laugh).

319 Opettaja: Joo-o (Antti ääntelee o-ouu, Reijo naurahtaa)
Yeaa-ah. (Antti says uh-h, Reijo gives a short laugh).

320 Pekka: Se on sanottu fysiikan kirjassa semmonen.
They said so in the physics book.

A learning opportunity was destroyed for all the students to construct the idea, that the instantaneous velocity is the gradient of the tangent. The group was offered a good start by Juha's idea, which he constructed by himself. But the other boys did not listen to him. Instead they tried to apply their knowledge from physics and got lost in that way.

The two episodes 7 and 8 show how students applied previously taught methods and principles by intuition and by remembering, without considering their applicability in those new situations. It is obvious, that this kind of approach leads to irrelevant acting in the situations of investigative studying, where new concepts, principles and methods are to be created, and it hinders students from having learning opportunities.

6.1.2 Teacher-students interactions

Sometimes I acted in a way compatible with my beliefs about the importance of students' own authentic thinking. When I really listened to the students I could adjust my talking with them so that it was connected to their experiences and thus made sense to them. This way many learning opportunities occurred to the students as well as to me. The assertiveness of Veikko, Mika and Juha also contributed so that it forced me to listen to the students' thinking.

Graph 7 The graph of the derivative function g´(x) of g(x)

Episode 9

The students in group B were considering the connections between a derivative function and its original function during the last small-group session (problem 6). In my discussion with the group I had figured out that the students had concluded that when the gradient of a tangent is positive, the original function is an increasing function and vice versa. Now they were claiming to me that the first and the second questions on the worksheet are the same. To a certain extent I accepted their point of view. Then the students said that the third question, where they were required to construct the graph of a third order polynomial function from the graph of its derivative function (graph 7) had nothing to do with the first two questions. They had solved the last one by constructing an equation for the parabola and by reversing the differentiation process, i.e. by integrating the expression. In the midst of an intensive discussion with Mika, I start explaining the connection.

49 Opettaja: Ja siinä on, oikeastaan minusta siinon kolme aluetta. (osoittaa sormellaan kysymyspaperia) Siinon niinkö, siinon niinkö tämä, tätä a:ta aikasempi, aikasemmat x:n arvot. Sitten väli a:sta b:hen ja sitten b:stä eteenpäin.
Teacher: And there are, actually I think there are three parts. (points with her finger to the question paper) There are, well, there is this, this previous, the x-values before a. Then, there is the interval from a to b and then forwards from b.

50 Mika: Kai sen niinkin voi [epäselvää].
Well, I guess you can [indistinct].

51 Opettaja: Nimittäin, jos katot tätä derivaattafunktion kuvaajaa. Mitä, ku sanoit sitä, että derivaatta on positiivinen tai derivaatta on negatiivinen (näyttää kädellään ilmaan nousevan ja laskevan suoran).
Because, if you look at the graph of the derivative function. What, you said that the derivative is positive or the derivative is negative (shows with her hand in the air a line sloping up and another sloping down).

52 Mika: Ym. (hiljaisuus)
Um. (silence)

53 Mika: Niihän ne on täsäki (osoittaa kynällään laskintaan).
But, so they are here as well (points with his pen to the calculator).

54 Opettaja: Niin on joo, ja miten se liittyy sitte siihen ite funktion kuvaajaan? Miten se
Well yes, and how is this connected to the graph of the function itself, then? How is it

55 Mika: Se kasvaa se funktio.
It increases there, the function.

56 Opettaja: Ni, sillä välillä kasvaa. Eli te ootta, ai, minä ihan ihmettelen miten te
Yeah, in that interval it is increasing. So you have, oh, I am wondering how you

57 Mika: Joo, se siis niinku, joo. Ko me aateltiin, että tämä kasvaa (osoittaa kynällään paraabelin kasvavaa osuutta) ja vaan se täsä, et se on positiivinen (näyttää kädellään kuin x-akselin yläpuolella olevaa käyrää). Joo.
Well, it, so yes. Coz we were thinking that this is increasing (points with his pen to the part of the parabola where the function is increasing), and that here it is positive (shows with his hand like a graph above the x-axis). Yes.

58 Opettaja: Niin, tä, nämähän on, tämä on nyt se kulmakerroinfunktio eli tästä voi päätellä eri x:n arvoille, mikä on se kulmakerroin tangentilla (osoittaa sormellaan kuvaajaa kysymyspaperissa).
So, this, these are, this is now the gradient function which means that you can deduce for different x-values what is the gradient of the tangent (points with her finger to the graph on the worksheet).

59 Mika: Ym.
Um.

60 Juha: Yhm (äänensävy osoittaa ymmärtämistä).
Um-m (the intonation shows understanding).

61 Mika: Joo ja tämä on täsä negatiivinen (osoittaa kysymyspaperin paraabelin vasemman puoleista negatiivista osaa) ja se laskee siinä (näyttää laskimestaan kolmannen asteen käyrän vasemman puoleista laskevaa osaa, Juha katsoo Mikan laskinta). Misä tämä ois negatiivinen (osoittaa kysymyspaperin paraabelin oikean puoleista negatiivista osaa), niin sillonki se on laskeva (näyttää vastaavaa kohtaa 3. asteen funktion kuvaajalta laskimesta).
Yes, and this here is negative (points to the negative part of the parabola on the left) and it decreases there (points to the decreasing part of the cubic polynomial on the left part of the screen of the calculator, Juha is looking at Mika's calculator). Where this is negative (points to the negative part of the parabola on the right), also there it is decreasing (shows the corresponding part on the graph of the cubic polynomial function in the calculator).

62 Juha: Ym. [epäelvää] Tuosson tuo positiivinen sitte, tuosson tuo.
Um. [indistinct] There it is positive, then, there it is

63 Mika: Ym. Aateltiin, väärin päin tämä ku [epäselvää] (näyttää kynällään kysymyspaperia).
Um. We thought wrong about it, because [indistinct] (points with his pen to the worksheet).

In turn 51, I am referring to what the boys had already concluded by themselves. Mika sees the connection (53) and soon he notices by himself what I am trying to say (57 – 61). Mika expresses his thinking and a learning opportunity occurs to Mika and Juha. The latter was following the discussion with attention. But during my discussion with Mika I had developed an interpretation that there still was a missing link in the conceptions of the students which hindered thorough understanding of the connection. I explained the idea to them that the values of the gradient function are gradients of tangents to the original graph (58). It seems that Mika ignores my message. But few minutes later the same idea occurs to his mind. And he starts explaining it with enthusiasm to the other students. So in this discussion, which unfortunately was rather unique between me and this group, rich learning opportunities occurred to the participating students.

But often I had problems in listening to the students thinking. I was either not very skilled or very accustomed to really trying to hear and understand what they were expressing while speaking to me. Often I had my own agenda for meeting the group which I wanted to realize.

Episode 10

The students in group A were working with the first derivative investigation (problem 4). They had been considering the meaning of the expression (f(z) – f(1))/(z – 1) for a time-distance graph.

64 Veikko: (opettajalle, naurahtaa) Niin mitä me nyt päätettiinkään? (Jenni seuraa keskustelua) Niin, kun kulmakerroin kasvaa, niin, niin tietenki sillon. No miten me se nyt äsken pähkäiltiin? Siis silleen, että
(to the teacher, laughs) Well, what did we decided? (Jenni follows the discussion) Yes, when the gradient increases, then, well, of course, then. How did we just think about it? Well, so that

65 Anni: Sen sekantin kulmakerroin kasvaa, niin ... (nauraa).
The gradient of the chord increases, then ... (laughs).

66 Veikko: Niin, että se, sitä
Yes, that the

67 Anni: pitemmältä matkalta
longer the distance

68 Veikko: ko se menee tietenkin jyrkemmin, niin se menee sinne, jos vaikka se on 10 otetaan se x:n arvoksi, 10 sehän nousee tietenki jyrkemin.
because it's steeper, of course. So it goes there, if it is for example 10, let's take 10 as the x-value, 10 it increases faster, of course.

69 Opettaja: Ym.
Um.

70 Veikko: Ja sit se kohtaa täällä jossakin sen 10 arvon. (näyttää käsillään sekantin kohtaamista käyrän kanssa).
And then it meets the value 10 around here. (shows with his hands how the secant meets the curve).

71 Opettaja: Joo.
Yes.

72 Veikko: Eli. Johonki tulokseen me äsken päästiin, mut se nyt, näät sen sitte tuolta filmiltä (kaikki nauravat).
So. We just came to a conclusion, but now. You'll see it in the film (everyone laughs).

73 Opettaja: Mutta mitä se sekantin kulmakerroin tarkottaa? (Hiljaisuus, Veikko ja Anni naurahtavat)
But what does the gradient of the chord mean? (Silence, Veikko and Anni give a short laugh.)

74 Opettaja: Onko teillä mittään aavistusta? Tai jotaki.
Have you got any idea? Or something.

75 Veikko: Ei mulla ainakaan oo mielessä mittään.
At least I don't have a clue.

76 Opettaja: (osoittaa Jennin laskimen nnäytössä olevaa lauseketta) Katoppako jos ensin annetaan z:lle arvo 4, niin, niin tuota täällä on $f(4) - f(1)$. $f(4)$:hän oli se kuinka pitkälle vaunu on kulkenu 4 sekunnissa.
(points to the expression on Jenni's calculator) Look, if we first give value 4 for z, then, well, then here we have $f(4) - f(1)$*. Namely,* $f(4)$ *tells how far the glider has travelled in 4 seconds.*

77 Veikko: Aivan.
Exactly.

78 Opettaja: Niin mitä on sillon $f(4) - f(1)$?
So, what's $f(4) - f(1)$ *then?*

79 Veikko: Sillä [epäselvää] alkunopeus. Elikkä siis $f(3)$, $f(3)$ siitä tulee. (Anni vilkaisee hymyillen Veikkoon.)
It has [indistinct] initial speed. Well, then $f(3)$*, it makes* $f(3)$*. (Anni smiles and looks at Veikko.)*

80 Opettaja: Eiku mikä on $f(1)$? Mikä on $f(1)$ sitte?
No, but what is $f(1)$*? What is* $f(1)$ *then?*

81 Veikko: Nii eikse oo se ihan alun. Eikö siis se on liikkunu sekunnin.
Well, isn't it … in the very beginning. No, it has moved one second.

82 Opettaja: Niin, kuinka pitkälle se on liikkunu sekunnissa.
Yes, how far it has travelled in a second.

83 Veikko: Niin.
Yes.

84 Opettaja: Nytkö otetaan $f(4) - f(1)$, niin mitä se tarkottaa?
Well, if we now think about $f(4) - f(1)$*, so what does that mean?*

85 Anni: Kuinka pitkälle se on siinä niitten vä, eikö kuinka pitkän matkan se on siinä niitten välillä kulkenut.
How far it has … in betw, I mean how long a distance it has travelled there between.

86 Opettaja: Edenny. Yhen ja, ajanhetken 1 viiva 4 sekuntia välillä kuinka pitkästi se on edenny sillä. No sitte jakajana on 4 - 1, se aika. Mitä tuo sekantin kulmakerroin on?
Travelled. Between one and, between one and four seconds, how far it has travelled there between. Well, then, the divisor is 4 – 1, the time. What is, then, the gradient of the chord?

87 Veikko: Siis se keskiarvo. Siis, tai ei siis keskiarvo, mut siis niinkö että. No mulla ei nyt oikein aivot pelaa.
So, it's the average. Well, not the average, but it is. My brains are not functioning at the moment.

88 Opettaja: Niin, hei, siinä on se kuljettu matka sillä välillä
Yes, look, here is the distance travelled in the interval

89 Veikko: Ja sitte se aika.
And the time.

90 Opettaja: Ni.
Yeah.

91 Veikko: Niin aika ja matka ja saahan keskinopeus.
So time and distance and we get the average speed.

92 Opettaja: Niin on. Se on keskinopeus on se sekantin kulmakerroin.
Yes it is! It's the average velocity, the gradient of the chord.

93 Veikko: (vähän hämmentyneenä) Ni. Ai jaa. (Opettaja nauraa. Opiskelijat kirjoittavat ylös.)
(looks astonished) Yes. Oh yeah? (Teacher laughs. Students write.)

The students have arrived to a conclusion that the expression (f(z) – f(x))/(z – x) is the gradient of the chord and that the greater the gradient is, the longer the corresponding time interval and the steeper the chord (64 – 72). This is the meaning that the students have given to the gradient. I don't see this, because it is different from the official meaning that I have in my mind. And I don't see a way to continue from what the students have constructed by themselves and to try to combine it with the tradition of mathematics. I decide to impose my own meaning (73). By asking simple questions as hints I lead the group to "find" a conclusion that I wish them to achieve, but which they were not able to reach by themselves. This kind of interactional pattern is called the Topaz effect (Brousseau, 1984, 1997, Novotna and Hospesova, 2007). In most cases the use of the strategy is accompanied by lowering of intellectual demands on students (Novotna and Hospesova, 2007). In this episode, even though Veikko states himself (91) that the expression represents average velocity, he is astonished (93) when I approve his conclusion and emphasize its importance. I interpret that the students did not express their own thinking. They tried to figure out what I wanted them to say. In the following episodes the students don't use the meaning average velocity for the expression (f(z) – f(1))/(z – 1) by themselves. Although Veikko and Anni actively participated in the discussion, I have to conclude that other than superficial learning opportunities were restricted for the students. This episode was one of the problematic ones for my criteria for determining whether a learning opportunity occurred or not.

6.2 Acting sociomathematcial norms and the occurrence of learning opportunities

There were some new sociomathematical norms negotiated and produced in my data, which better suited the investigative approach and also contributed to the occurrence of learning opportunities. On the other hand, especially in the beginning of the experimental course, sticking to many traditional sociomathematical norms in Finnish upper secondary mathematics hindered the occurrence of learning opportunities in the new situation.

Creative and profound investigating

In the beginning of the course the students were accustomed to the practice that a mathematical task is solved by using known procedures and that an answer to a mathematical task is normally one number or expression. During the few first sessions in group A we were negotiating the norm: *When investigating mathematics, one should try to approach the topic in a profound and creative way.*

The following extracts describe a process during the three first investigative sessions. I am labeling all of them together as Episode 11 (problems 1 - 3).

Episode 11

When investigating the limit of the function sinx/x at x = -1 the students in group A first gave an answer that when x approaches zero from the right or the left the values of the function increase. I had to prompt them to observe more exactly. At the end of the session Anni constructed and the others agreed that the values of the function approach one when x approaches zero from the right or from the left. But, when writing their summary, they started talking only about the increasing of the function. Again I had to ask them about the way the values of the function increase.

When working with the second investigation, where the values of the function increase to infinity or decrease to minus infinity when x approaches a certain number, the students give some attention to how the values of the function increase or decrease. But in their notes they only write about the increasing and decreasing of the values of the function. When I come to see the group, I read the notes and again ask the students to be more exact. The students explicate to me that the values of the function increase in an explosive way. During the third investigation about limits at infinity and at minus infinity, the students first talk only about the increasing and decreasing of the functions, but soon they continue without my prompting to investigate the particular ways the values of the function increase or decrease.

98 Veikko: Ei, mutta ei se ikinä saavuta sitä, ykköstä pienemmäksi mee.
No, but it never gets there, it never goes smaller than one.

99 Anni: Mennee se ku se [epäselvää].
It does, when it [indistinct].

100 Veikko: [epäselvää] (Anni naurahtaa)
[indistinct] (Anni gives a short laugh)

101 Anni: Tai, en mie oikeastaan tiiä. Mie oon yks pilkku nolla, nolla, nolla ykkösessä. (naurahtaa) Ei se pienene enempää.
Or, I am not quite sure. I am at one point zero, zero, zero one. (laughs) It doesn't get any smaller.

102 Veikko: Mitä?
What?

103 Anni: Ei se pienene enempää.
It doesn't get any smaller.

104 Veikko: Niin se ei mee niinkö sen ykkösen alle. Sen takia ko tuosson tuo miinus ykkönen.
So it doesn't, like, go smaller than one. Because there is this minus one.

105 Anni: Ei, mennee se. Oota. En mie tiiä vielä. Se jää siihen ykköseen.
No but, it does. Wait a minute. I don't know yet. It stays at one.

106 Veikko: Jääkö?
Does it?

107 Anni: Jää. Ko se on, esimerkiksi viiessäsaassa niin se on vielä yks. Miten ois tuhat? Joo. Se on ykkönen.
Yes it does. Coz it is, for example, at five hundred it's still one. What about one thousand? Yes. It's one.

108 Veikko: Vai niin. [epäeselvää]
Okay. [indistinct]

109 Anni: Miten
How

110 Veikko: Eihän ne saavuta sitä, ehkä. [epäselvää] (naurahtaa)
I don't think they'll reach it. Maybe. [indistinct] (gives a short laugh)

111 Anni: Ym. Ym.
Um. Um.

112 Veikko: Hyvä ku huomasit (Annikin naurahtaa)! Miten se siis muotoillaan, muotoillaan nyt sanallisesti?
Luckily you noticed (Anni laughs)! How shall we put it in words?

113 Anni:	Sillonku se lähenee tuota miinus ykköstä tuolta negatiiviselta puolelta, niin se kasvaa. Mihin tuo kasvaa? (katsoo laskimensa näyttöä) Kutoseen. Ei. (naurahtaa) Sen näkkee tuolta alhaalta. (naurahtaa) Sitteku on nolla, niin se arvo on nolla. *When it approaches this minus one from the negative side, then it is increasing. Up to what? (looks at the screen of his calculator) Up to six. No. (gives a short laugh) You can see it there below. At zero the value is zero.*
114 Veikko:	Niin. (Anni naurahtaa) Mut siis miten me laitetaan se, että se ei saavuta ikinä sitä miinus ykköstä. Tai, tai siis pyssyy siinä ykkösessä siis. *Yes. (Anni gives a short laugh) But, how shall we put it, that it never reaches minus one. Or, I mean, stays at one.*
115 Anni:	No laita *Well, put*
116 Veikko:	Ko seki taitaa olla, se taitaa olla jonkunlainen raja-arvo kans. Et se ei mee ykkösen alas. Voisko se olla? *Because it may be, it may also be some kind of limit. That it doesn't go below one. Could it be?*
117 Anni:	Niin. *Yes.*
118 Veikko:	x-akselin suuntainen *In the direction of the x-axis.*
119 Anni:	(naurahtaa)Raja-arvo. *(laughs a bit) Limit.*
120 Veikko:	Niin. *Yes.*

Anni and Veikko together find that the values of the function $\frac{x^2}{x^2-1}$ "don't decrease below 1" and construct that there might be a "limit parallel to the x-axis". They both have a learning opportunity. Jenni did not seem to be listening to the conversation at this moment. When the students in my data took a more profound and creative approach to the problems, learning opportunities occurred for them. On the other hand, satisfaction with superficial or repetitive answers destroyed learning opportunities.

Different methods

On some occasions, at the beginning of the experimental course, the students tried to act according to the traditional sociomathematical norms: *An accepted solution for a mathematical task must be in symbolic form.* and *There are certain rules for solution methods and for writing down the solution and the answer to a mathematical task.*

Episode 12

Juha, Pekka and Reijo in group B were beginning the third investigation about limits at infinity (problem 3). They were first asked to construct a sequence of x-values that increase above all boundaries and then a second sequence, the numbers of which decrease below any boundaries. Mika starts by writing down a simple sequence.

8 Mika: No, eikö se oo niinkö että yks, kaks, kolome, nelijä (Juha nauraa, samoin Pekka) Nehän on varmasti, kasvaa
Well, isn't it that one, two, three, four (Juha laughs and Pekka, too) They do increase.

9 Juha: Mika rupia kirijottamaan vaan semmosia.
Mika, you start writing those numbers.

10 Mika: Mitä siinä on kuule. Kasvaa ne yli rajojen. Ym.
Listen to me, guys. They do increase above all boundaries. Yes.

11 Juha: Eikö se oo sama niinku kirijottaa näin (piirtää ääretön merkin Mikan monisteeseen).
You could as well write this (draws the symbol for infinity).

12 Mika: Niin. Ei se nyt heti ihan sama asia oo. No mitä sillon väliä. [epäselvää] Kirjotetaan yks, kymmenen, kakskymmentä, viiskymmentä, sata, jne.
Yes. But it's not exactly the same. What does it matter? [indistinct] Let's write one, ten, twenty, fifty, hundred and so on.

13 Juha: Ää, ää
Aa, aa.

14 Mika: No, eikö ne kasva loputtomiin? Valitettavasti.
Well, they increase forever. Unfortunately.

15 Pekka: Niin justiin. Niin vois laittaa vaikka puoli, yks, yks ja puoli, kaks (Juha ryhtyy selailemaan oppikirjaa), kaks ja puoli, kolme, kolme ja puoli
Exactly. You can put half, one, one and a half, two (Juha starts leafing through his textbook), two and a half, three, three and a half

16 Mika: Nih-ii.
Yeaa-ah.

17 Juha: Kaikki desimaalit, kymmenen tuhannen desimaalin tarkkuuella.
All the decimal places, to the accuracy of ten thousand.

18 Mika: Niin, kasvaa ne äärettömyyksiin. (naurahtaa, samoin Juha, Pekka hymyilee) [epäselvää]
Yes, they increase to infinities. (laughs, so does Juha, Pekka smiles) [indistinct]

19 Pekka: (ryhtyy kirjoittamaan) Niin, pistetään vaikka että
(starts writing) Yes. Shall we put that

20 Juha: (lukee monisteesta) Muodosta sellainen pienenee, x:n arvojen jono, jonka jäsenet pienenevät rajattomasti. (selaa oppikirjaa)
(reads from the worksheet) Construct such, decreases, a sequence of x-values, the terms of which decrease below all boundaries. (leafing through his textbook)

21 Mika: Kyllä ne kasvaa yli rajojen.
They do increase above boundaries.

22 Pekka: Pienenee [epäselvää] koko ajan. (Mika ja Pekka kirjoittavat, Juha selaa kirjaa)
Decreasing [indistinct] all the time. (Mika and Pekka write, Juha is leafing through the textbook)

23 Mika: Ei sitä piä niinkö liian monimutkasesti.
You don’t need to make it too complicated.

Juha is mocking Mika about his simple and concrete sequence (8 – 9). He is suggesting that the symbol for infinity would be a good answer (11). Mika elaborates his sequence and justifies it (14, 18 and 21). He expresses the belief that, when investigating mathematics, working with concrete numbers is acceptable (23). Juha rejects the solutions offered by Mika and starts to leaf through his textbook. I interpret that Mika and Pekka have a learning opportunity to construct what it means that numbers increase above all boundaries, but that Juha did not. He did not arrive at a solution. Sticking to the overemphasis of symbolic representation destroyed a learning opportunity for him. On the other hand, by accepting the use of numerical representations, Mika contributed to the learning opportunity he and Pekka had.

Sticking to the old norms about method and presentation of answers destroyed learning opportunities in my data. On the other hand, giving up the overemphasis of symbolic representation and symbolic methods and acting according to the norm: W*hen investigating mathematics, different approaches, in addition to the symbolic methods are approved* offered learning opportunities to the students.

Symbolic and increasingly more abstract concepts and methods is the nature of academic mathematics. The Finnish upper secondary high level mathematics courses should prepare the students for university level studies in mathematics, science and engineering (Opetushallitus, 1994). Here, as well as in the assessment policy of the national matriculation examination board, are the roots for the appreciation of symbolic representation. But in traditional teaching in Finnish upper secondary mathematics, symbolizing is done by the teacher and it often happens too quickly for many students. I am claiming, with Repo (1996), that average and low achieving students don’t have the opportunity for constructing rich personal meanings for mathematical concepts. Mathematics becomes, to a certain extent, meaningless manipulation of symbols and procedures, a “game” to be played with certain rules.

Procedures instead of properties of mathematical objects

Because of the tasks used in the investigations during the experimental course, we often talked about mathematical objects and mathematical reality in our discussions. Thus we were producing, to some extent, the culture of inquiry mathematics. But on the other hand, old norms and values were strongly part of our doing mathematics. Focusing on procedures instead of mathematical objects is close to the "instructions" metaphor. During the last investigation about the relationships of a derivative function and its original function, the students had a choice, at least in principle, between a conceptual approach and a procedural approach.

After discussing the concepts of derivative and derivative function, and finding the few first derivative functions through the difference quotient, I had justified and taught to the students how to differentiate polynomial functions. The rules and exercises were easy and all the students were able to achieve the necessary skills. But after this, the meaning of the concept of derivative function to the students was that it is the result of the differentiation process.

Episode 13

6	Reijo:	Mikä on derivaattafunktio? *What is a derivative function?*
7	Mika:	No se on just se, ko ottaa pois sen (ryhtyy selaamaan vihkoaan). *Well, it's the, when you take away the (starts leafing through his notebook).*
8	Juha:	(ynisee suklaata suussaan) *(murmurs with chocolate in his mouth)*
9	Reijo:	[epäselvää] *[indistinct]*
10	Mika:	Se mitä te Pekkan kanssa muka olitta niinku hulluina. Se, se on se derivaattafunktio, mikä sieltä tullee. Ko x kolomanteen oli kak, kolome x toiseen. *It's what you were crazily doing with Pekka. It's, it is a derivative function which you get from there. When x cubed became tw, three x squared.*
11	Reijo:	Ai niin. *Oh, yes.*

The boys continue by constructing an expression for a downwards opening parabola with two zeros and use an inverse process for differentiation to find the equation of the third order polynomial. They draw the graph of the function with their calculators.

The boys did remember that a derivative function is the result of the differentiation process. My goal, when giving the task to the students, was to have them construct the idea that we can deduce about the increasing and decreasing of a function by looking at whether the derivative function has positive or negative values, because the values of the derivative function are gradients of tangents to the original function. It was my mistake to allow the procedure type solution by my choice of the function. But in the solution processes of both of the groups it was obvious that they appreciated the procedural approach to the task instead of trying to use and develop their conceptual knowledge for sketching the required graph. Both emphasis on procedures, and talk solely about solving tasks can be seen as opposite positions to stressing the construction of mathematical objects and mathematical reality. The learning opportunity to understand the connection between a function and its derivative function at a deep conceptual level was destroyed because of the over appreciation of procedures.

A right answer implies a right method

Episode 14

When group A was constructing the methods for finding the equations of a tangent and a normal to a curve (problem 5), the knowledge about the existence of the right answer from the calculator made them solve the problem about the tangent without completely understanding the process. When the students realized that they had enough numbers, they solved the problem, rejoiced about the right answer and continued with the investigation. But when writing their summary they started discussing the role of x_1 in the formula $y - y_1 = k(x - x_1)$.

83 Anni: Miten tuo x_1?
What is this x_1?

84 Veikko: Ja x_1:nen, mistä se tulee? No se tulee
And x_1, where does it come from? It comes from

85 Jenni: No sehän on se.
Well, it is it.

86 Veikko: Se on vaan tuo x:ä.
It's simply the x.

87 Anni: Ym.
Um.

A learning opportunity to understand the formula, and the roles of the symbols in it, at a deep level was diminished for the students who were content to find the right answer and trusted that it ensures that they have solved the problem in a right way.

Pang (2001) describes the microcultures of two classrooms, which had similar student-centered social norms but different sociomathematical norms. Among other features, he mentions that in the first classroom there was emphasis on getting the right answer and in the other classroom a correct answer without a mathematically justifiable process was rejected. Pang uses the first classroom as a warning example that simply changing classroom social norms is not enough for promoting conceptual learning or characteristically mathematical ways of knowing. Students acquire conceptual underpinnings of mathematics when they are participating in explanation, justification and argumentation that are specific to mathematical activity and discourse (Pang, 2001.) In the school mathematics tradition, getting the right answer easily removes the need for justifying the solution method for a problem in other ways and thus hinders students from gaining learning opportunities.

6.3 Summary

It is no surprise that acting according to the collaboration norm, expressing one's thinking and listening to the others as well as justifying one's claims, challenging others by disagreeing and asking questions contributed to the occurrence of learning opportunities. In addition to these qualities of peer interaction, if I, as a teacher, really listened to the students and communicated with them in an authentic way learning opportunities occurred for them.

When the students took a profound and creative approach to the topic, learning opportunities occurred for them. And accepting different methods, namely graphical, numerical, drawing and writing, for example, were also connected to richness of learning opportunities.

Breaching the collaboration norm by individualistic behavior or retirement as well as lack of justifying and disagreeing with others destroyed learning opportunities in the data. On the other hand, in group B challenging others sometimes turned to competition and disputes. The focus of the discussion switched to the social mode and mathematical learning opportunities were lost. Learning opportunities were also hindered when one or two members of the group were clearly more assertive than the others, especially if the others normally retired in the case of conflict. Some old roles of students from traditional teaching prevented them from having learning opportunities. For example, instead of constructing new knowledge, the students might apply the methods shown by the teacher during the previous lesson or they might insist on applying ready knowledge instead of creating new ideas. In the teacher-students interactions learning opportunities were sometimes destroyed when I tried to impose my own agenda on the meeting without really listening to the students' contributions.

In the situation of the investigative approach, sticking to the acceptance of solely symbolic methods and representations destroyed learning opportunities. Also an emphasis on procedures and instructions about how to solve tasks hindered learning opportunities to create meaning for the mathematical objects at hand. Acting according to the belief that a right answer implies a right method worked against the occurrence of learning opportunities.

7 DISCUSSION

7.1 Summary of the results

The “resistance” or “friction” which I felt during the experimental course was partly due to some traditional social and sociomathematical norms which were in contradiction with the new way of studying, and partly a consequence of the unconscious negotiation of some new norms. These new norms were most often connected to the investigative nature of the approach. Through asking my students to investigate mathematics in small-groups, I initiated the implicit negotiation of the norm: *In the investigative approach it is the role of the students to express their* ***own thinking***. Very often the students acted according to my expectations and thus produced the same norm. But my negotiation was not consistent. Occasionally I behaved so that my actions conveyed the belief: *It is the role of the students to find the official meanings of mathematical objects.* Many times the students also had difficulties in acting in a relevant way with the new approach. They were producing the traditional norms: *The role of the students is to solve tasks according to instructions given by the teacher.* or *The role of the students is to apply ready made knowledge.* New sociomathematical norms that needed to be negotiated in an explicit way were: *When investigating mathematics, one should approach the topic in a profound and creative way, When investigating mathematics, different approaches in addition to symbolic methods are appreciated.* and *When investigating mathematics, questions about method, accuracy and writing down the solution are situation dependent.*

It was clear in the data, that we were not accustomed to justifying our mathematical claims. In both peer and teacher-student interactions we accepted social rather than mathematical reasons for our claims. Production of two social norms was connected to this: *The acceptance of a mathematical claim is on the basis of social agreement.* and *The acceptance of a mathematical claim is on the authority of the speaker*. It seems, however, that the students had sound beliefs about what is a valid mathematical justification. In the rare cases, when we did justify our claims, we acted according to the traditional sociomathematical norm: *Explicit justification in mathematics must be based on the properties of mathematical objects.* The situation here is different than what Yackel and Cobb (1996) report about negotiating social and sociomathematical norms with young children (Yackel and Cobb, 1996). Because the students in my project were high level mathematics students in a Finnish upper secondary school, they seem to have a certain sense of mathematical rigor and of what is an appropriate mathematical justification although in our face-to-face interactions the norm for mathematical justification was not yet established.

In both of the groups the students were producing the collaboration norm: *When working in small-groups, students are expected to try to arrive at a consensual conclusion.* and the norms: *When working in small-groups, students* *are expected to* ***express*** *their own thinking.* and *In small-group discussions students are expected to listen to others.* But the right to speak and the obligation to listen were not based on equity of the students. In both of the groups, also in my presence, we produced the social norm: *Those who are more assertive than others have the right to express their thinking, and others have the obligation to listen to them.*

The participation of the students in the discussions in the two small-groups was different. Although in both of the groups the students were producing the norm: *When working in small-groups students are expected to show agreement*, in group B the students produced, to a greater extent than in other interactional make ups, the social norms: *In small-group discussions students are expected to show disagreement and challenge each other.* and *In small-group discussions students should justify their claims.* In the interactions of group A the students produced the norm: *In small-group discussions it is not appropriate to show disagreement too clearly.* Furthermore, in group A the social basis for the acceptance of mathematical claims was in the social agreement, but in group B it was in the authority of the speaker. All the assertive students in the two small groups who constantly interrupted others were males. The differences in the participation structures which I found in my small groups fit with observations about the different styles of interaction in all-male and all-female small-groups (De Vries et al., 2002, Keys, 1997, Tolmie and Howe, 1993). It may be that they reflect the sociolinguistic subcultures of girls and boys (Maltz and Borker, 1982).

The questions given to the students inspired talk about mathematical objects and this way I was initiating the implicit negotiation of the norm: *Mathematical discussion is about mathematical reality,* and the students were producing the norm. But, on the other hand, we often slipped into producing the norm: *Mathematical discussion is about solving tasks*. However, in their written text, the students produced the traditional norm in our school: *Written mathematical text is exact and unambiguous.* When the problem given to the students had one exact answer, we were producing the sociomathematical norm: *A right answer to a problem ensures that the method is right.*

Trying to achieve consensual conclusion, expressing one's thinking and listening to the others as well as disagreeing, asking questions and justifying one's claims, all contributed to the occurrence of learning opportunities. If I listened to the students and communicated with them in an authentic way, learning opportunities occurred. Taking a creative and profound approach to investigations and accepting different methods and representations also added to the number of learning opportunities.

Learning opportunities were destroyed by breaching the collaboration norm as well as by lack of disagreement and justifying. On the other hand, especially in group B, challenging others often turned to competition and disputes. The focus of the discussion turned to a social mode and mathematical learning opportunities were lost. Two old roles of students from traditional school mathematics, namely following instructions given by the teacher and applying ready-made knowledge, worked against the occurrence of learning opportunities.

If I imposed my own agenda without really listening to the students, learning opportunities were lost. Sticking to symbolic methods, emphasis on procedures and instructions and the belief that the right answer ensures the right method, worked against the occurrence of learning opportunities.

7.2 Elaborations of the emergent perspective

7.2.1 Critique of the perspective

I find the important concepts of the emergent perspective, namely, social norms, sociomathematical norms, mathematical practices, the corresponding beliefs and conceptions and learning opportunities, highly relevant in describing the nature of interactions and conceptual learning when small-groups are investigating mathematics. But analyses conducted in the emergent perspective typically take only the practices of the local community as a point of reference (Cobb and Yackel, 1996). My results raise questions about the role of other cultural processes, coming from outside the school, in the negotiation and production of classroom social and sociomathematical norms. For example, the sociolinguistic subcultures of girls and boys in my data were part of producing social norms for small-group discussions in mathematics.

In the teaching experiment of my thesis the students were allowed to choose their partners for the small-group activity for themselves. Almost all groups consisted of girls or boys only. Gender of peers seemed to be of particular importance to the students in that context. While writing the transcripts for the original four small groups, I noticed the different styles of interaction in the male and female groups. I have written two papers (Partanen, 2005, 2007) abut the styles of linguistic peer interaction of girls and boys in those small groups. The categories of gender were not imposed on my data, instead there is evidence that they were important to the students and that they were constantly produced by them.

The differences in the styles of interaction in the two small groups of this research are highly in line with Maltz and Borker’s theory (1982) about sociolinguistic subcultures of girls and boys. Other researchers have reported similar findings. Studies by Lindroos (1997) and Staberg (1994) support the view that, in Scandinavian classrooms, boys are more assertive than girls. Brown and Gilligan (1993) document several cases of private school girls, who linked argument with losing relationships. Lindow, Wilkinson and Peterson (1985) noticed that among second and third graders the means for solving a conflict varied with achievement and gender. They also observed that boys had more attempts at explaining one’s position and had more answers accepted by other participants than girls. In their research review on the use of small-groups in science instruction, Bennet et al. (2010) report on three studies (De Vries et al., 2002, Keys, 1997, Tolmie and Howe, 1993) which identified clear differences in interactional styles of all-male and all-female small-groups when upper secondary students were discussing questions in science. In the all-male groups the students tended to confront differences in their individual perspectives, but in the all-female groups the students searched for common features of their predictions and explanations in order to avoid conflict.

Seen together with Maltz and Borker's theory (1982) and the findings of the other researchers, my results raise the question about whether norms for participating in small-group discussions are gendered. It may be that when working in small-groups, students find the context of friendship groups relevant for them, and production of social norms for small-group discussions in mathematics lessons is intertwined with cultural processes coming from the surrounding society.

Gorgorio and Planas (Planas and Gorgorio, 2004, Gorgorio and Planas, 2005a, 2005b) arrived at a similar critique of the emergent perspective in their research on immigrant students' transition processes in Spain. They show, how in a classroom with immigrant and native students, immigrant students were not expected to explain their mathematical ideas, but to show their algorithmic processes. On the contrary, native students were given the challenge to explain, discuss and argue. The immigrant students were expected to listen to other students' explanations. They were expected also to use real contexts but not necessarily to relate them to mathematics. Gorgorio and Planas (2005b) report that "it became apparent that the construct of norms did not fully allow us to interpret what we were observing: different understandings of the same norm within a mathematics classroom were difficult to reconcile, and could certainly not be taken as shared" (p. 3/65 – 3/72).

Another example of the insufficiency of the classroom perspective concerns the norms: *When investigating mathematics, different approaches in addition to symbolic methods are approved.* and *When investigating mathematics, questions about method, accuracy and writing down the solution are situation dependent*. During the experimental course the students utilized their symbolic calculators and used its numerical, graphical and symbolic functions. While solving the tasks and problems the students also drew diagrams and wrote texts. As a Finnish upper secondary mathematics teacher I was worried whether the negotiation of these norms would harm my students' success in mathematics examinations later in their studies. The national matriculation examination board is the only examination board in the country. Through writing questions and through their assessment policy, they guide mathematics teaching in Finnish upper secondary schools. Especially in high level mathematics examinations, they emphasize symbolic methods in solving tasks. Normally the use of numerical or graphical approaches is penalized. Students also will lose marks for wrong types of answers. The renegotiation of the "different methods" norm seems to be vital for the investigative approach and students' conceptual learning. But, it was in contradiction with the normative expectations at the national level of schooling.

The concepts of classroom social and sociomathematical norms were developed in the course of an ongoing program of developmental research in which instructional design and classroom-based research went hand in hand. Cobb and Yackel deny that their interpretive framework might, in some sense, capture the essence of psychological and sociological processes in the classroom. Instead, they take a pragmatic view and emphasize that often the explanations referring solely to classroom processes were adequate for their particular purposes (Cobb and Yackel, 1996.) It may be that, if the main goal of an intervention is to build an inquiry mathematics culture and show its benefits, the concept of classroom norms is sufficient in helping to describe and guide the process. As a teacher researcher and according to my methodology, I have a different standpoint than research groups going into schools, mainly for research interests. In more naturalistic settings it may be necessary to recognize that production of normative participation in a mathematics classroom may be intertwined with or influenced by cultural processes. Research that focuses on understanding the renegotiation of norms initiated by ordinary teachers in their classrooms and schools is needed to support teachers in this demanding task. To be able to negotiate appropriate norms efficiently, I argue, teachers need firstly to understand the initial situation, including its relevant connections to outside the classroom, and secondly to understand aspects of the negotiation process.

7.2.2 Elaborations of the framework

Relational framework by Cobb and Hodge

Cobb and Hodge (2002) review a wide range of research in the field and present a relational framework, which extends the analyses conducted in their design research and corresponding teaching experiments to take account of issues of cultural diversity and equity. They conceptualize diversity "in terms of students' participation in the practices of either local, home communities or of broader groups within wider society" (p. 252). The authors emphasize that, instead of using imposed standard categories of race and ethnicity, the students' own views of themselves and others should be taken seriously. The categories used in analyses of classroom events should be based on the actual lived identities of the students.

For describing the participation of students in the practices of their local home communities Cobb and Hodge (2002) adopt the concept of a community of practice (Lave, 1991, Lave and Wenger, 1991). For their own purposes they reduce Lave and Wenger's (1998, p. 125 – 126) fourteen indicators that define a community of practice to three interrelated dimensions that serve to differentiate it from less closely related groups of people: a) mutual engagement, b) negotiated enterprise and c) a repertoire of negotiable resources that have accumulated over time. A community of practice is not static, but evolves over time. It is the local context of its members' learning. Individual's learning is seen to occur when they participate and contribute to the evolution of the communal practices and norms.

This perspective on local home communities highlights students' learning outside the classroom (Cobb and Hodge, 2002.)

Cobb and Hodge (2002) use the concept of discourse (Gutierrez et al., 1999, Gee, 1997) to characterize the broader communities of which students are part. Discourses are sociohistorical coordinations of people, objects, interacting etc. which allow for the display and recognition of socially significant identities like being a feminist, a street gang member or a theoretical physicist for example (Gee, 1997). Cobb and Hodge (2002) clarify that communities of practice create cultures and in doing so bring to the fore their members' mutual engagement in joint activities. But in viewing broader communities beyond the communities of practice as generating a discourse, they bring to the fore communication, interaction and brokering between representatives of different communities of practice. However, only as we participate in local communities, can discourses touch our experience (Cobb and Hodge, 2002.)

According to Cobb and Hodge (2002), their relational perspective does not reduce diversity to the out-of-school ways of reasoning and talking. Diversity for them is about the continuities and discontinuities between those ways of acting and the culture of the mathematics classroom. Diversity is located "at the nexus of students' lives in classrooms and their participation in the practices of a) broader communities in wider society and b) the local home community" (p. 256).

The methodological implications of the relational perspective do not include that researchers should collect data about students' lives outside the school. Rut rather it means that an interpretive stance is taken to students' activity at school. The focus is on the mathematics classroom (Cobb and Hodge, 2002).

Gorgorio and Planas

Gorgorio and Planas (2005b) explain their path to reconstructing norms. They started their research project with the constructs of social and sociomathematical norms established by Cobb and his colleagues. However, with the development of their study of immigrant students' transition processes they became convinced that the construct of norms did not fully allow them to interpret what they saw. Finally they arrived at the decision to adopt a sociocultural perspective and to revisit the word *social* in the social norms and sociomathematical norms. "From a sociocultural perspective the learning of mathematics is affected by what takes place within the classroom and in their nearest contexts. We could not understand any more the word social as simply ´being conjointly constructed by the different participants in the classroom´, without considering that all participants were, in turn, social individuals, with their own social and cultural experiences and expectations" (p. 67).

Gorgorio and Planas (2005b) define sociomathematical norms as the explicit or implicit regulations that influence participation within the mathematics classroom and the interactive structure of the development of the mathematical practice. They are connected to questions about how the different participants value mathematical knowledge, and position and value themselves and others in relation to mathematical practices and knowledge. Sociomathematical norms of Gorgorio and Planas are very close to the concept of social norms of Cobb et al. (Gorgorio and Planas, 2005b.)

Norms of mathematical practice of Gorgorio and Planas, on the other hand, are norms that legitimate mathematical activity, strategies, processes and certain ways of thinking. They have to do with the rules and ways of doing mathematics as a scientific discipline and as a school subject. When teachers decide what is appropriate in school mathematics they borrow their meanings from the culture of the groups of which they are part. They also borrow their meanings from the culture of the educational system and from their particular school cultures. Students interpret what mathematics is about through the lens of the culture(s) they have participated in during their lives, whether it is the classroom(s) culture(s), the school(s) culture(s) or their home culture. Norms of mathematical practice are close to Cobb and Yackel's (1996) sociomathematical norms (Gorgorio and Planas, 2005b.)

Gorgorio and Planas (2005b) conclude by stating that for them, norms are secondary cultural artefacts as defined by Cole (1996). Norms mediate classroom interaction by dealing with questions about who (appropriately) participates, whose participation is (not) welcomed, and the different roles played by individuals within the mathematics conversation. They emphasize the interplay between cultural scripts and social representations.

Commenting on the elaborations

I found the concepts of the emergent perspective relevant and very helpful in describing interaction and learning in my data. But the problems of discussing other cultural processes, which are connected to the negotiation and production of norms and practices in the classroom, bring forth the need for broadening the perspective. Cobb and Hodge (2002, 2007) as well as Gorgorio and Planas (2005b) turned to socio-cultural theorizing when dealing with these problems of the emergent perspective. I am confused by the decision of Cobb and Hodge to include ingredients from both the emergent perspective and sociocultural theorizing in their relational framework. It may be that after the classic paper on the emergent perspective (Cobb and Yackel, 1996), Cobb's discussion of the framework has been inspired more and more by socio-cultural theory. But I am used to thinking that social constructivism and socio-cultural theorizing stem from different theoretical foundations. My thinking is influenced by Lerman (1996) and Confrey (1995) who argue that the Piagetian and Vygotskian programs have fundamentally different orientations, like different world views, and there can be no resolution of them. According to Lerman (1996) a merger of these two views would be incoherent, and it can only be attempted by not engaging fully with their distinct interpretations of the individual in her or his actions in the world. Confrey (1995) argues that an interaction between the two strands constitutes a significant change in both theories, and will require a theory that is neither Piagetian nor Vygotskyan, but draws heavily on both. I see a certain compatibility in the background theories of the emergent perspective (Cobb and Yackel, 1996): radical constructivism, symbolic interactionism and ethnomethodlogy (see the discussion in chapter 2). But seen, at least from the fundamentalist socio-cultural perspective, the integration of social processes into the epistemological framework of radical constructivism is impossible (Lerman, 1996, Waschescio, 1998). Gorgorio and Planas (2005b) changed their course totally and they started working inside sociocultural theorizing. Because of that they had to redefine the concepts of social and sociomathematical norms.

Concerning the problems raised by my analysis, I don't find the complementarity of the emergent and sociocultural perspectives presented by Cobb and Yackel (1996) or the relational framework of Cobb and Hodge (2002, 2007) completely satisfactory positions, although the latter includes very interesting and inspiring ideas. The conflict of the use of different methods when investigating mathematics and the normative expectations at the national level, could possibly be explained by taking into account the national context of schooling (Cobb and Yackel, 1996) or seen as my participation in two different discourses (Cobb and Hodge, 2002, 2007), those of a reformer in mathematics education and a Finnish upper secondary mathematics teacher. But I don't see the sociolinguistic subcultures of girls and boys in my data as communities of practice. They reflect the identities of the students as female and male teenagers and they could possibly be understood as discourses (Cobb and Hodge, 2002, 2007). Nevertheless, I think there still is a need for theoretical and empirical investigations which could clarify the connections between developing the culture of a mathematics classroom and the wider cultures (school culture/s and the cultures of the wider society) in relevant ways.

I wonder whether further theorizing stemming from the traditions of symbolic interactionism and ethnomethodolgy could contribute to broadening the emergent perspective. Both of the traditions are rather general approaches (Jungwirth, 1996). Jungwirth (1996), for example, sketches a framework for the research on gender and mathematics based on symbolic interactionism and ethnomethodlogy. She writes about the methodological implications of such an approach and stresses the importance of taking into account 1) that the meaning of a thing depends on the context and 2) is tied to the process it emerges from. With Cobb and Hodge (2002, 2007) I emphasize that when taking into account cultural diversity, the focus should be on categories which are based on students' and teachers' own views of themselves and the situations. Standard categories of race and ethnicity should not be imposed without considering their relevance. In the discussion about elaborating the emergent perspective the following concepts might, among others, bear fruit: context (Erickson and Schultz, 1981), social context (Mehan et al., 1976, Erickson and Schultz, 1981), cultural context as well as social competence (Hymes, 1974). Attempting to extend the emergent perspective by starting from its original theoretical roots might be a challenging aim for further research.

7.3 About teaching and learning mathematics and developing the use of the investigative small-group approach

Before my study there was plenty of evidence that working in an active way with mathematical problems in a CAS environment promotes students conceptual learning, and that algorithmic skills may last longer if students have good conceptual understanding (Repo, 1996). For this reason I didn't find it sensible to compare the investigative small-group approach with traditional teaching in my research. Instead, I decided to focus on developing the use of the approach. It is obvious, on the basis of my results, that new social and sociomathematical norms need to be negotiated for the investigative small-group approach. The method did not neatly fit with the traditional classroom and work is needed for its potential to be realized. On the other hand, beginning to use the approach made visible some unintended consequences of traditional mathematics instruction. In this way the investigative small-group approach challenged the school mathematics culture of which I am part as well. In this section, I shall discuss how I could develop the use of the approach in my teaching. Many of the ideas are also most probably relevant for other upper secondary teachers who strive for the same goals. In addition, I shall discuss aspects of teaching and learning mathematics in general and give some suggestions for further research.

In our face-to-face interactions, we were not accustomed to justifying our claims mathematically. Instead, the acceptance of ideas was made on a social basis. This was so, although my findings suggest that the students in the two small groups had some kind of idea of what is a valid mathematical reason. I think that in Finnish schools the authority of the teacher as the knower is very strongly established. Furthermore, teaching mathematical argumentation is not part of our school mathematics tradition (Kaasila et al., 2010). The pattern of the more knowledgeable to be an authority is easily translated into the peer-interactions of the students (Good et al., 1992). Another phenomenon working against mathematical justification is that there may be groups and students who find disagreeing difficult and threatening (Lampert et al., 1996) and who, for that reason, rather search for common features in their thinking (De Vries et al., 2002, Keys, 1997, Tolmie and Howe, 1993).

It seems to me that the norm about justification on a mathematical basis is connected to other norms compatible with the inquiry mathematics tradition. A good situation would be established by problems or investigations where students are likely to construct different solutions and mathematical ideas as Gravemeijer (1994, 1997) suggests. The tasks in my project were not planned that way. Then, students should express their own ideas to others, who should listen and try to understand. The students should show agreement and, more important, disagreement. If the goal of the students is to try to achieve consensual conclusion in the group, the need for justifications is created. As a teacher, I could model for the students the use of mathematical justifications. In my face-to-face interactions with the students, I should not use my authority but justify my claims in a relevant way to the students. I should require a mathematical justification when a student expresses a statement. I could also take students' relevant contributions as examples and thus negotiate the mathematical justification norm. One should also try to create an accepting atmosphere in the classroom so that students are not afraid of being wrong.

I am not quite sure whether all these actions are enough for changing the culture of doing mathematics in my classrooms in Lyseonpuiston lukio. Normally we teach the same students for six and a half weeks only, after which the classes are reformed. Negotiation of new norms in the whole school with hundreds of students would require many mathematics teachers. It is much harder than just trying to change the culture of your own classroom. An interesting question for further research would be: How can mathematical argumentation be established and supported in Finnish upper secondary mathematics schools and classes?

Another big problem which was illuminated by my research is that democratic collaboration in the small-groups was not spontaneous. Three research reviews on the use of small-group discussions in instruction conclude that the approach normally promotes students' mathematical learning and acquisition of high order skills (Good et al., 1992, Kumpulainen, 2002, Bennet et al., 2010). But, they also suggest that domination of some students, most often high-achievers, and retirement of others is a common feature in the interactions of many small-groups. My results chapters describe situations where profound learning opportunities were destroyed by assertive students who did not listen to another student but took the floor for themselves. An important aim for future research and developmental work would be to shed light on how democratic relationships among the members of small-groups could be promoted and thus the potential of small-group approaches be better realized. Esmonde (2009) found that the type of assignment given to the students for small-group work was connected to the equity of participation. Richmond and Striley (1996) and Kurth et al. (2002) stress the need for a leader to adopt an inclusive style and share tasks equitably around a group. In their research, this promoted more substantial engagement in the discussions by a number of participants and increased the quality of the discussions. The norms of listening to others, trying to understand their thinking and that of justification on a mathematical basis might be the most important norms in establishing equity of participation. However, the problem of domination seems to be so common and its effects on the fruitfulness of collaboration so crucial that the question needs to be examined more thoroughly.

An interesting point on the quality of mathematical discussions in lessons is to what extent they reflect "acting in and elaborating a taken-as-shared mathematical reality" (Cobb and Yackel, 1998, p. 163) rather than procedural instructions. Sfard (2000) describes how mathematical discourse and mathematical objects create each other. If we aim for students' construction of mathematical concepts, we should talk about the concepts as if they existed and as if we acted on and with them. To some extent the tasks used in my experimental course guided the discussions in this direction. But the production of the corresponding norm was not consistent. Since the teaching experiment, I have started observing my own talk and writing when teaching. It is more natural for me to talk in ways which convey that what we are really doing in the classroom is learning how to solve mathematical tasks. I wonder if this is one important reason for the poor conceptual learning in Finnish upper secondary mathematics classes (Haapasalo et al.,1995, 1997, Merenluoto, 2001). Tasks for the investigative small-group approach should be planned so that they guide and require talking about mathematical objects and the norm of explanations and mathematical talk to be about mathematical reality should be established.

The use of graphical and numerical approaches to mathematical problems, as well as drawing and writing, in addition to the symbolic method can be seen to enhance possibilities for students to construct rich understandings of mathematical concepts (Repo, 1996, Park et al., 2007). It is a pity that the Finnish matriculation examination board has taken so strict a stance against other than symbolic methods and against calculators which make these possible. Because the investigative small-group approach is in some ways in contradiction with the tradition of teaching mathematics in upper secondary schools, especially the sociomathematical norms compatible with it, I recommend that the small-group work and following whole class discussion should be attempted to be done a context of its own. New norms should be negotiated and established for that working method only. I think it is possible that upper secondary students can learn to differentiate between the contexts of the investigative small-group approach and solving tasks during normal lessons and exams. On the other hand, transition of certain norms, especially those concerning communication and argumentation, would not be a negative phenomenon. They would improve the quality of discussions and doing mathematics during all phases of a mathematics lesson. Anyway, I believe there will be some transition, but the negotiation and anti-negotiation should be intensive concerning those norms which might be harmful for the students in the examinations.

My research questions concerned what kind of social and sociomathematical norms were negotiated or produced in our interactions. After my research project I have a vision of what kind of norms should be negotiated in a situation similar to mine so that the potentials of the approach could be better realized (see section 7.6). To support teachers in their attempts to construct inquiry mathematics cultures in their classrooms, attention should also be paid to those processes by which norms can be negotiated and are negotiated in efficient ways. This would be a very interesting aim for further research.

Change from traditional teaching to applying constructivist principles may be difficult for a teacher (Richardson and Placier, 2001). Most successful in changing teachers' beliefs, conceptions and practices in this direction have been long-term, collaborative and inquiry-oriented programs (Richardson, Placier, 2001, Cobb, Wood and Yackel, 1990). My negotiation of social and sociomathematical norms was not consistent in the data. I think one reason was that during the experimental course I was not aware of the importance of norms. But on the other hand it also reflects my temporary state of confusion and my being in the process of change. When transcribing and analyzing my data I have had wonderful opportunities for reflection, and writing the research report has helped me in connecting my experiences with theoretical knowledge in the field. During the years I have taught the basics of calculus by the investigative small-group approach several times. I feel much more confident with it now because I can better understand what is involved. And I have goals for my students and for my own development. But the confusion, deep processing of your beliefs and the first steps of doing things differently may be difficult and threatening, especially since the traditional norms are so well established in schools. In Finnish upper secondary schools, students choose their schedule from a choice of courses and if the school is big enough they have options of choosing among teachers as well. In my class there were students who strongly opposed the new approach which they saw as my own pedagogy (although it was highly compatible with the contemporary syllabus), and some of them were high-achievers. They would have liked me to teach the theory and examples and give them exercises afterwards. This was a very difficult experience for me because it created a threat of boycott to my courses and thus a threat to my teacher's identity (Krzywacki, 2009). There was a moment during the experimental course when I would have liked to give up had I not been collecting data for my research project. Through all this I want to emphasize the urgent need of teachers for reflection and support when striving towards changing their teaching in this direction.

7.4 Critical considerations of my research project

For Cobb and Yackel (1996, 1998) the context of the social and sociomathematical norms is the mathematics classroom. They developed the important concepts of the emergent perspective for situations where new norms needed to be negotiated for the realistic mathematics approach. In their perspective, explicit negotiation of norms is emphasized. My use of the concepts of social and sociomathemtical norms is not that simple and elegant. I needed different ideas about norms and their genesis to be able to describe with satisfaction the ecology of norms in my situation and my data. There were the traditional norms for doing mathematics in our school. In addition, germs of new norms could be seen. But it was not possible for me to try to show the existence of new norms, because at the time of the data collection it was too early for that. I decided to analyze the norms that were being negotiated in an implicit or an explicit way. In addition to this I also needed the idea of production of norms from the interactionist and ethnomethodology traditions. I have the discussions of the two small groups as my data. Through the interactions in them I made interpretations about classroom norms, but sometimes I saw the norms negotiated and produced to be specific for the small-group phase of the mathematics lesson.

The interest in my research on social and sociomathematical norms was not in contributing to the recent norms research (McClain and Cobb, 2001, Ju and

Kwon, 2007, Rasmussen and Stephan, 2008) which investigates new established norms, their relationships and is developing methodology for analysis. During the teaching experiment I didn't even know I was going to study norms in my data and thus I was not able to negotiate appropriate norms for the approach. The strength of my research is in the application of the norms research to a naturalistic setting when a teacher starts using the investigative small-group approach in her teaching. I think my study has brought the concepts of social and sociomathematical norms closer to the context of Finnish upper secondary mathematics instruction than the design experiments done in different countries could do. During recent years investigations have appeared in Finnish upper secondary mathematics textbooks. I hope to have produced useful knowledge and rich ideas for those teachers who start using them.

There was a strong interpretive element in my analysis. According to my paradigm I used my personality and my teacher's experience in analyzing and constructing the results and descriptions. Someone else would have done the analysis of the same data in a different way and arrived at different conclusions. Thus my results can by no means be seen as objective scientific knowledge. This is obvious to an educated reader, but I think that sometimes in the public and the media the difference may disappear. My results are neither all-encompassing nor normative. But I think they can function as a stimulus for reflection and further investigations. And they make a good starting point for me, and perhaps other mathematics teachers, in further developmental work.

7.5 Teacher empowerment through research

During my research process I was inspired by Kincheloe's (1991) ideas about critical constructivist teacher research as a path to empowerment. In this section I am going to reflect my own experiences and conceptions about domination in my career and my empowerment through the research project. Empowerment here is to be understood in the context of critical theories.

It seems to me that the two worlds I am part of, namely, community of mathematics teachers and the group of researchers in mathematics education, are quite separate. Many mathematics teachers like to talk about the uselessness of educational knowledge to them. What they enjoy is discussions with other mathematics teachers who understand their practice and important questions that emerge from it. On the other hand, I have given critical comments to some researchers who seemed to talk about teachers as an ignorant and resistant herd. It is not actually many researchers who emphasize the importance of taking into account teachers' points of view when doing research. This seems to be the situation at least in mathematics and science education in my country. There are some teachers doing research, thanks to the possibilities offered by new graduate schools, but projects where researchers go to schools and develop instruction together with teachers are rare. The separation still exists, at least to some extent.

Because of the situation, teachers are tied or restricted in their reflection on their work to the world of practice. Some in-service teacher training courses are being offered, but in the present economical situation it is harder for teachers to participate in them. And very often new theoretical movements or ideas are transformed before they meet teachers because they have been going through the thinking of many different persons: researchers, teacher educators, lecturers and headmasters, for example. They may even be used for opposite purposes than the original theorists emphasized as has happened with action research (Kincheloe, 1991).

Access to original insights of theoretical ideas about education is one important aspect of my empowerment through the research project. Being able to consider my practice and all kinds of educational phenomena through different theoretical “lenses” has enriched my reflection and understandings. For example, before my PhD studies I tended to think about what happened in schools in psychological terms only. I might feel great pressure if something went wrong in my teaching and blame myself. Now I understand that there are different perspectives to schooling: administrative, societal, political and pedagogical, for example, and that the classroom is embedded in the school culture and the culture of the society. Students are members of their families and different communities. Students’ behaviour and attitudes can be seen to be connected to other things in addition to my influence on them. I have learnt to try to recognize what are my responsibilities and what are not. Before my present studies I could feel frustration when someone with authority expressed ideas about schooling and instruction which were in contradiction with my own philosophy. Now I am able to situate those different ideas, as well as mine, in theoretical and historical contexts. And, furthermore, I am able to evaluate and criticize them. Feminist theorizing has given me tools for understanding about myself as a female mathematics teacher in my working place, and in society, and it has caused a revolution in my thinking about knowledge and knowing.

I am passionate about continuous learning and about developing my personality and teacher’s practice. For a while I have been interested in the use of the investigative small-group approach in my upper secondary classes. I think that the most important aspect of my empowerment through the PhD project has been in my learning to connect theoretical and practical knowledge in my developmental work, and in my acquisition of dispositions of a researcher. I have grown to take charge of my personal development and my developmental work. I am no more dependent on occasional teacher training courses or controlled by anybody else in that. Helpful skills here are the abilities of finding knowledge about educational research and studying it critically.

During the numerous seminars and workshops where my research has been discussed and criticized, I have learnt something indispensable. To a certain extent my original teacher training strengthened in my mind the illusion that there are good teachers and bad teachers and you are born to either of these categories. This may be due to some charismatic teachers in the school where we were trained or due to the evaluation system of pre-service teachers of that time. I am claiming that it is often difficult for us as teachers to receive critical comments on our work. Through the seminars I have met all kinds of criticism, relevant and irrelevant. I have grown to tolerate it, and what is more important, I have learnt that the criticism may help me to broaden my perspectives and improve the quality of my work. I have noticed that this disposition has been transferred to different projects where I have been working with people in other contexts.

During the PhD studies my self confidence has increased. This is seen especially in my increased courage to think for myself and be critical even to ideas presented by appreciated scholars. For this I have to thank especially the courses organized by the Nordic Graduate School in Mathematics Education. In a course about quantitative methods in educational research we, led by the lecturer, studied the extent to which there were deficiencies in the use of statistics in many articles in respected journals. One task in a course on research methodology was to pick an article on mathematics education and try to critically evaluate its quality and present our ideas to others. This was a starting point for my growth in courage.

There were hard times for me in the world of research also. Being a woman, a teacher and a research student caused some male senior researchers to think they had the right to underestimate my plans for methodology and theory as well as my writings, for the only reason that they were not in line with their own ideas about doing educational research. A seminar group was finished forever after I, as a beginning research student, had given a presentation about my plans to use an ethnographic approach. The reason was the “bad quality of the research done in the group”. However, reflecting on this kind of occasion has made me even stronger and added to my self confidence.

7.6 Scheme for negotiating social and sociomathematical norms for the investigative small-group approach

For further developmental work I have constructed a scheme in which I can initiate negotiating social and sociomathematical norms compatible with the investigative small-group approach for an initial situation similar to mine. To be exact, the scheme is valid for my data only, but I think it also makes a good starting point for further developing my own practice and, perhaps, for other upper secondary mathematics teachers in Finland who wish to use the method in their teaching. The scheme presented here is not an order. But it includes relevant focus points when trying to negotiate suitable norms for the investigative small-group approach.

Preparations

The potential of the approach becomes best realized if planning of instruction, organization of classroom and negotiation of norms go hand-in-hand. The teacher should design the assignments and problems or apply those made by others for her students. The type of tasks and group composition are connected to the ways of participation and equity of students in the group work. The teacher should try to create a safe and non-competitive atmosphere. She should give a suitable introduction to the students for the investigation. After the small-group session the teacher should lead the whole class discussion about the topics of the investigation.

To begin with

Beginning from the very first experiments with the investigative small-group approach the following norms should be negotiated or strengthened:

- When working in small-groups, the students are expected to try to arrive at a consensual conclusion.
- In the investigative approach the role of the students is to express their own thinking and create new mathematics. Their role is not to search for ready-made knowledge.
- When investigating mathematics, one should approach the topic in a profound and creative way.
- When investigating mathematics, different approaches in addition to symbolic methods are appreciated: numerical and graphical methods, drawing and writing, for example.
- The mathematical activity is investigating mathematical reality and objects, not learning how to solve tasks. Solutions to problems should be explained in terms of those objects.

Mathematical argumentation and equity of participation

When suitable, the teacher should initiate the negotiation of norms concerning mathematical argumentation in the small-groups.

- Students are expected to express their thinking, to listen to everybody who wants to comment and try to understand their points. (The initial preferred styles of interaction of girls and boys in peer groups may differ in this respect.)
- Students should ask for explanations and give one when someone requires it.
- Students should express agreement and disagreement. (Students using linguistic strategies typical of girls may find it difficult to disagree too clearly.)

- Authority should not be used as justification, but mathematical reasons for claims should be given.
- A right answer is not sufficient justification of the solution method. Valid argumentation is.

Teacher's role

- The teacher should listen to students' authentic thinking and *based on that* answer questions, comment or teach new ideas.
- The teacher should guide students' construction of mathematical meaning in the direction of the tradition of mathematics. She can do this by asking questions or giving comments and in the whole class discussion she can pick students' relevant contributions to the topic and build on them.
- The teacher should initiate the negotiation of social and sociomathematical norms.

A teacher can start the negotiation of new norms in explicit and in implicit ways. The best results are most probably obtained when both ways are used and are aligned. It is then up to the students whether they accept the challenge and change their behavior and beliefs or not. Implicit negotiation can be done, for example, by giving certain types of assignments and tools to the students or just asking them to work in a different way. The teacher can model the expected behavior for the students. But it is also important to initiate the negotiation of norms in an explicit way. For example, before a session you can tell the students what is expected of them or afterwards you can point to some positive occurrences in the classroom. During the working of small-groups the teacher can make comments to the students about their behavior. The teacher should try to be consistent and patient in her negotiation. She should not give up too easily! And she should trust her students.

7.7 Reforming mathematics classrooms

Researchers and teachers have developed many promising approaches for teaching and learning mathematics. Realistic mathematics education (Treffers, 1987, Gravemeijer, 1997, Gravemeijer and Doorman, 1999), investigations (Ernest, 1991, Lerman, 1989 and Morgan, 1996), small-group discussions (Good et al., 1992, Bennet et al., 2010) and the use of CAS programs in learning mathematical concepts (Repo, 1996, Heid, 1988, Berry and Newman, 2003) are just a few examples to be mentioned. But as a Finnish upper secondary teacher I want to pose a critical question. How much of this wonderful research and developmental work, even with evidence of better learning outcomes, has really touched students' lives in mathematics classrooms? In his PhD project Pasi Sahlberg (1996) arranged teacher training courses for reforming instruction in

Finnish schools. Although he took a systemic approach and viewed schools as capable of learning and self organization, he arrived at a very pessimistic view about the possibilities of changing school cultures.

With Sahlberg (1996), I emphasize that teachers are the key people for making an educational change. But, too often, I have seen cases where my colleagues, myself included, learn about a new teaching method, try it in our instruction and, after a while, give it up. The logic of these experiments always seems to be the same. The learning outcomes achieved by the new approach are compared to those obtained by traditional teaching, or students are asked whether they like the new approach or not. Comparing the learning outcomes is done by intuition or by results in the course examinations. No extra data is collected and analyzed. These kinds of comparisons are not fair, and they prohibit teachers from developing new approaches. Very often it is hard to find convincing evidence about better learning outcomes through the new method, because the outcomes may be different from those produced by traditional teaching. On the other hand, teachers have developed the skills needed in traditional teaching for years and both teachers and students know what to expect of the others and what are expected of them. A more fruitful attitude, when adopting a new innovation, would be to start with the idea that we are educating our students for their future studies and careers. In the future world, students need experiences of different learning environments. They need skills for collaboration and flexibility. This kind of rationale for constructing teaching experiments helps in giving the time and space needed for new approaches to be developed until their potential is realized. I am dreaming of a school where developing new practices and teaching methods, with experience and patience, is part of the everyday life!

Through my research, however, I have come to understand that applying new approaches in naturalistic settings is a job on its own, as hard as the original developmental work. There is no shortcut to success, here. But, I have also learnt that *it is possible* for you to change instruction in schools, especially if you are a teacher. We need to know about new innovations, construct them ourselves or let them arise from our practice. We need understandings of developmental work in the context of a school, what kinds of aspects are involved and are important in the change. In addition, on the basis of my experiences about the ease of giving up, I see that very often a certain kind of agreed commitment is needed for really making an impact. A planned project with researchers or other teachers, or just with yourself, as well as someone responsible for steering the project until its goals are achieved, would make a structure which forces you to go on. We need to be open-minded. In addition to changing our practices, the new approaches may require changing our deep beliefs (Richardson and Placier, 2001), epistemological orientations (Sahlberg, 1996) or our values. Reflection in the form of discussions with others, reading literature or writing is necessary. A teacher planning developmental work should think about organizing a supportive network for herself. You deserve it. Indispensible dispositions are friendliness, respect for others, patience and persistence. If you are ready to work hard, this kind of developmental work at schools is a rich way of living with plenty of satisfaction. It is a professional way of working as a teacher (Kincheloe, 1991).

YHTEENVETO

Teoreettinen viitekehys

Sosiaalisen konstruktivismin kenttään sijoittuva emergentti viitekehys (emergent perspective) (Cobb ja Yackel, 1996, 1998) ammentaa juurensa symbolisesta interaktionismista (Blumer, 1996), etnometodologiasta (Mehan ja Wood, 1975) ja radikaalista konstruktivismista (von Glasersfeld, 1989, 1995a). Sen perusperiaatteena on, että koululuokan vuorovaikutuksessa psykologisen ja sosiaalisen tason ilmiöt ovat refleksiivisessä suhteessa toisiinsa. Yksittäiset opiskelijat vaikuttavat aktiivisesti luokkayhteisön mikrokulttuuriin, joka on samalla se ympäristö, jossa he henkilökohtaisesti toimivat ja oppivat. Opiskelijan matemaattista käyttäytymistä ei voida ymmärtää irrallaan luokkayhteisön kulttuurista, joka toisaalta koostuu ja syntyy (emerges) juuri sen jäsenten vuorovaikutuksesta.

Cobb ja Yackel (1996) ovat kehittäneet tulkintakehyksen (interpretive framework) (figure 1, s. 27), jonka avulla voidaan koordinoida psykologisen ja kollektiivisen tason analyyseja. Interaktionistista näkökulmaa käytetään analysoitaessa sosiologisia ilmiöitä kuten *luokkayhteisön sosiaalisia normeja* sekä *sosiomatemaattisia normeja*. Edellisen psykologisia vastineita ovat yksilöiden *uskomukset omasta roolistaan, muiden rooleista sekä matemaattisen toiminnan luonteesta*. Jälkimmäiset syntyvät opiskelijoiden ja opettajan yhteensopivista *matemaattisista uskomuksista ja arvoista*. Kolmas kollektiivisen tason ulottuvuus Cobbin ja Yackelin (1996) tulkintakehyksessä on *luokkayhteisön matemaattiset toiminnot* (classroom mathematical practices). Aivan samoin kuin tarkastellaan yhden opiskelijan oppimista, voidaan myös analysoida koko oppilasryhmän matemaattista kehitystä. Esimerkiksi tietyssä vaiheessa ala-asteella joistakin tavoista tulkita lukusanoja ja numeroita tulee luokkayhteisössä itsestään selviä, jolloin niitä voidaan kutsua luokkayhteisön matemaattisiksi toiminnoiksi. Näiden psykologisia vastineita ovat yksilön *matemaattiset käsitykset* ja *toiminta*. Emergentin viitekehyksen mukaan luokkayhteisön alati kehittyvät matemaattiset toiminnot, tarkasteltuna sen sosiaalisia ja sosiomatemaattisia normeja vasten, muodostavan sen ajallisen ja paikallisen tilanteen, jossa yksittäinen opiskelija matemaattisesti kehittyy. Opiskelijan oppiminen ei kuitenkaan ole suora seuraus hänen osallistumisestaan luokkayhteisön matemaattisiin toimintoihin, vaan nämä luovat edellytyksiä matemaattiselle oppimiselle (Cobb ja Yackel, 1996.)

Olen tutkimuksessani soveltanut Cobbin ja Yackelin (1996) emergenttiä viitekehystä. Sen kehittäjät korostavat tutkivalle matematiikan opiskelulle suotuisten normien eksplisiittisen neuvottelun tärkeyttä. Analyysissani käytin myös aineksia interaktionistisesta teoriasta (Bauersfeld, Krummheuer and Voigt 1988, Voigt, 1994). Sitä soveltavat tutkijat ovat kiinnostuneita luokkahuoneen vuorovaikutuksen implisiittisistä säännöistä, jotka syntyvät rutiineista ja niihin liittyvistä velvoitteista. Tutkimukseni tavoitteena oli antaa kuvaus sosiaalisista ja sosiomatemaattisista normeista, jotka olivat läsnä kokeilukurssillamme. Tähän tavoitteeseen pyrin analysoimalla millaisia normeja kahden pienryhmän vuorovaikutuksessa neuvoteltiin ja tuotettiin. Sen lisäksi halusin valottaa miten normien mukainen toiminta oli yhteydessä oppimismahdollisuuksien syntymiseen. Haasteena teorioiden soveltamisessa oli, että luokkahuoneen normit kuvaavat koko opetusryhmän mikrokulttuuria, mutta aineistoni käsitti nauhoituksia vain kahden pienryhmän työskentelystä. Olen ikään kuin katsonut luokkahuoneeseen näiden kahden pienryhmän kautta. Toinen vaikeus syntyi siitä, että opetuskokeiluni kesti vain vähän aikaa, enkä sen aikana tietoisesti neuvotellut normeja. Uudet syntymässä olleet normit eivät vielä olleet vakiintuneet. Lisäksi perinteiset normit olivat taustalla vahvasti koko ajan. En voinut ottaa tavoitteekseni osoittaa tiettyjen normien olemassaoloa. Sen sijaan päätin tehdä tulkintoja siitä millaisia normeja pienryhmien vuorovaikutuksessa neuvoteltiin ja tuotettiin.

Metodologia

Tutkimukseni liittyy ”opettaja oman työnsä tutkijana”-perinteeseen (Stenhouse, 1975, Elliot, 1988, Cochran-Smith ja Lytle, 1999, Altrichter et al., 2008). Sen lisäksi, että tavoitteena oli tuottaa uutta tietoa matematiikan oppimisesta ja opettamisesta, yhtä tärkeänä päämääränä oli kehittää tutkivan pienryhmätyöskentelyn käyttöä opetuksessani. Näiden päämäärien kautta halusin myös kasvaa opettajana ja ihmisenä. Tutkimuskysymykset nousivat omasta työstäni kun otin käyttöön uuden ja erilaisen työmuodon. Aineiston analysointi ja raportin kirjoittaminen ovat antaneet minulle mahdollisuuden syvään reflektointiin ja oman opettajuuteni prosessointiin. Teoria antoi minulle relevantit käsitteet, joiden kautta tarkastelin omaa opetustani. Tuloksillani on toivottavasti merkitystä tutkimusalallemme, ja rakentamani kehys normien neuvottelua varten antaa hyvän pohjan jatkossa tapahtuvalle kehitystyölle.

Tyypillistä tutkimukselleni on etnografinen kokonaisote (Spindler ja Spindler, 1992, Tedlock, 2000, Hammersley ja Atkinson, 1983). Tutkimusongelmat nousivat aineistosta ja kokemuksistani opetuskokeilun aikana. Ne ovat jatkuvasti tarkentuneet, ja olen sovittanut niitä yhteen teorian kanssa. Analyysissani sovelsin Cobbin (1995) kehittämää menetelmää, joka on lähellä mikroetnografista vuorovaikutusanalyysia. Videonauhoitusten ja litteroidun puheen avulla analysoin tarkasti ja yksityiskohtaisesti opiskelijoiden puhetta ja vuorovaikutusta, mutta tehdessäni tulkintoja asetin aineiston ilmiöt laajempiin yhteyksiin. Tilanteiden ja koulukulttuurimme tuntemus sekä oma persoonani tulivat mukaan tulkintoihin.

Analyysimenetelmä

Analyysivaiheen aluksi jaoin litteroidun aineiston episodeihin keskustelun teeman mukaan. Kustakin episodista tarkastelin 1) millainen oli ryhmän osallistumisrakenne, 2) millaisia matemaattisia uskomuksia ja arvoja ryhmässä ilmaistiin, 3) millainen oli kollektiivinen merkitystenantoprosessi, 4) syntyikö opiskelijoille oppimismahdollisuuksia ja 5) mitä opiskelijat oppivat. Muodostin kriteerit sen arvioimiseksi, milloin tietylle opiskelijalle syntyi oppimismahdollisuus. Analysoin episodit aikajärjestyksessä, kumpikin ryhmä erikseen.

Jo episodien analysoinnin aikana aloin kehittää hypoteeseja ryhmälle tyypillisestä osallistumisrakenteesta. Kirjoitin ne muistiin. Uusien episodien pohjalta arvioin hypoteesejani yhä uudelleen ja uudelleen. Näin kehittyi lopulta sellainen kuvaus ryhmälle tyypillisestä vuorovaikutuksesta, johon saatoin olla tyytyväinen. Sen pohjalta tein johtopäätöksiä siitä, millaisia sosiaalisia normeja ryhmässä neuvoteltiin ja tuotettiin. Matemaattisten uskomusten ja arvojen analysointi oli helpompaa. Kun olin käynyt läpi kaikki ryhmän episodit, luokittelin tilanteet ja tein johtopäätökset neuvotelluista ja tuotetuista sosiomatemaattisista normeista.

Analyysin loppuvaiheessa tarkastelin syntyneitä oppimismahdollisuuksia. Kussakin tilanteessa, jossa opiskelijalle oli syntynyt oppimismahdollisuus, analysoin, millaisten normien mukainen toiminta oli myötävaikuttamassa siihen. Jos selvä oppimismahdollisuus jostakin syystä estyi, kuvasin myös niissä tilanteissa, millaisten normien mukainen toiminta oli läsnä. Lopuksi luokittelin ne vuorovaikutuksen lajit, jotka olivat joko edistäneet tai ehkäisseet oppimismahdollisuuksien syntymistä.

Tuloksia

Kokeilukurssin aikana kokemani ”kitka” tai ”vastus” johtui osittain perinteisistä sosiaalisista ja sosiomatemaattisista normeista, jotka olivat ristiriidassa uuden lähestymistavan kanssa sekä osittain eräiden uusien normien tiedostamattomasta neuvottelusta. Nämä uudet normit liittyivät useinmiten opetusmenetelmän tutkivaan luonteeseen. Antaessani opiskelijoille tutkimustehtäviä ratkaistavaksi, aloitin implisiittisen neuvottelun koskien normia: *Tutkivassa opiskelussa opiskelijoiden rooli on ilmaista omaa ajatteluaan ja luoda uutta matematiikkaa.* Usein opiskelijat toimivatkin odotusteni mukaisesti ja osallistuivat täten kyseisen normin tuottamiseen. Mutta minä itse en ollut aina johdonmukainen toiminnassani ja neuvottelussani. Toisinaan käyttäytymiseni viesti uskomuksesta: *Tutkivassa opiskelussa opiskelijoiden rooli on keksiä matemaattisten objektien ”oikeat” merkitykset.* Usein myös opiskelijoilla oli vaikeuksia tässä suhteessa. He saattoivat toimia perinteisten normien mukaisesti: *Opiskelijan rooli on ratkaista tehtäviä opettajan antamien ohjeiden mukaisesti.* tai *Opiskelijoiden rooli on soveltaa valmista tietoa.* Aineistossani eksplisiittisesti neuvoteltuja uusia sosiomatemaattisia normeja olivat: *Tutkittaessa matematiikkaa ongelmia tulee* *lähestyä syvällisesti ja luovasti, Tutkittaessa matematiikkaa erilaiset menetelmät symbolisten menetelmien lisäksi ovat hyväksyttäviä.* ja *Tutkittaessa matematiikkaa ei ole selviä sääntöjä tarkkuudesta ja ratkaisun esittämistavasta.*

Tulokseni osoittavat selvästi, että matemaattisissa keskusteluissamme emme olleet tottuneet perustelemaan väitteitämme. Sekä vertais- että opettaja-opiskelijavuorovaikutuksessa väitteitä hyväksyttiin ja hylättiin ennemminkin sosiaalisista syistä. Kaksi näin tuottamaamme normis ovat: *Matemaattiset väitteet hyväksytään yksimielisyyden perusteella.* ja *Matemaattiset väitteet hyväksytään puhujan auktoriteetin perusteella.* Analyysini tulokset viittaavat kuitenkin siihen, että pienryhmien opiskelijoilla oli terve näkemys siitä millainen matemaattinen perustelu on pätevä. Niinä harvoina kertoina, kun perustelimme väitteitämme, tuotimme yleensä normia: *Eksplisiittinen perustelu matematiikassa nojautuu matemaattisten objektien ominaisuuksiin.* Kuvattu tilanne on erilainen kuin Yackelin ja Cobin tutkimuksessa (1996) pienillä lapsilla. Tutkimukseni opiskelijat olivat suomalaisen lukion pitkän matematiikan opiskelijoita. Heille näyttäisi jo syntyneen jonkinmoinen käsitys matemaattisen tiedon luonteesta, vaikka emme olleetkaan tottuneet perustelemaan väitteitämme lähivuorovaikutuksessa oppitunneillamme. Molempien ryhmien toiminnassa tuotettiin normia: *Pienryhmätyöskentelyssä opiskelijoiden odotetaan pyrkivän yhteisymmärrykseen aiheesta,* sekä normeja: *Pienryhmätyöskentelyssä opiskelijoiden tulee ilmaista omia ajatuksiaan.* ja *Pienryhmätyöskentelyssä opiskelijoiden pitää kuunnella toisiaan.* Mutta puhumisen oikeus ja kuuntelemisen velvollisuus eivät jakautuneet tasaisesti. Molemmissa ryhmissä, myös minun läsnä ollessani, tuotimme sosiaalista normia: *Ne, jotka tuovat itseään esille aktiivisemmin kuin muut, saavat puhua ja muiden tehtävä on kuunnella.*

Tutkimani pienryhmät olivat erilaisia vuorovaikutustyyliltään. Vaikka kummassakin ryhmässä tuotettiin normia: *Työskenneltäessä pienryhmissä opiskelijoiden odotetaan pyrkivän yhteisymmärrykseen*, ryhmässä B tuotettiin enemmän kuin muissa kokoonpanoissa sosiaalisia normeja: *Pienryhmätyöskentelyssä opiskelijoiden odotetaan ilmaisevan erimielisyyksiä ja haastavan toisiaan.* ja *Pienryhmäkeskusteluissa opiskelijoiden odotetaan perustelevan väitteensä.* Ryhmässä A taas tuotettiin normia: *Pienryhmissä keskusteltaessa ei ole sopivaa ilmaista erimielisyyksiä liian selvästi.* Lisäksi tässä ryhmässä matemaattiset väitteet hyväksyttiin, kun keskustelijat olivat niistä samaa mieltä, kun taas ryhmässä B hyväksyminen tapahtui puhujan auktoriteetin perusteella. Kaikki itseään aktiivisesti esille tuovat opiskelijat, jotka myös keskeyttivät toisia puhujia, olivat poikia. Havaitsemani erot pienryhmien osallistumisrakenteissa sopivat yhteen tyttö- ja poikaryhmien vuorovaikutuksen tyyleistä tehtyjen muiden havaintojen kanssa (De Vries et al., 2002, Keys, 1997, Tolmie ja Howe, 1993). Voi olla, että ne heijastavat tyttöjen ja poikien erilaisia sosiolingvistisiä alakulttuureja (Maltz ja Borker, 1982).

Opiskelijoille antamani kysymykset synnyttivät keskustelua matemaattisista objekteista ja niiden kautta aloitin implisiittisen neuvottelun koskien normia: *Matemaattinen keskustelu käsittelee matemaattisen todellisuuden luonnetta*.

Mutta toisaalta meidän keskustelumme lipsahti usein puheeksi, jossa tuotimme normia: *Matemaattinen keskustelu käsittelee tehtävien ratkaisemista.* Kirjoittamassaan tekstissä opiskelijat toimivat normin *Kirjoitettu matemaattinen teksti on yksikäsitteistä ja täsmällistä.* mukaisesti. Kun opiskelijoille annettuun ongelmaan oli vain yksin oikea ratkaisu, tuotimme normia: *Oikea vastaus ongelmaan takaa sen, että ratkaisu on oikein.*

Yhteisymmärrykseen pyrkiminen, omien ajatusten ilmaiseminen ja toisten kuunteleminen, kuten myös erimielisyyden osoittaminen, kysyminen ja omien väitteiden perustelu edistivät oppimismahdollisuuksien syntymistä pienryhmissä. Jos minä aidosti kuuntelin opiskelijoita ja kommunikoin heidän kanssaan siltä pohjalta, syntyi oppimismahdollisuuksia. Se, että ongelmia lähestyttiin luovasti ja syvällisesti vaikutti positiivisesti oppimismahdollisuuksien syntymiseen.

Oppimismahdollisuuksia estyi, kun opiskelijat eivät toimineet yhteistyössä tai kun erimielisyyttä ei osoitettu eikä väitteitä perusteltu. Toisaalta erityisesti ryhmässä B toisten haastaminen johti usein opiskelijoiden väliseen kilpailuun ja väittelyihin. Keskustelu ei enää ollut luonteeltaan matemaattista, vaan sosiaaliset tarkoitusperät voittivat akateemisen kiinnostuksen. Näin tuhoutui monia oppimismahdollisuuksia. Kaksi perinteistä opiskelijoiden roolia ehkäisi oppimismahdollisuuksien syntymistä ryhmien opiskellessa uudella tavalla. Näin kävi, kun tutkimisen sijaan opiskelijat yrittivät seurata opettajan antamia ohjeita tai vain soveltaa valmista tietoa. Jos minä vieraillessani pienryhmässä en aidosti kuunnellut opiskelijoiden omia ajatuksia, vaan ajoin läpi oman katsomukseni heistä välittämättä, se ehkäisi oppimismahdollisuuksien syntymistä.

Lopuksi

Jo silloin, kun suunnittelin tutkimustani, oli runsaasti näyttöä siitä, että aktiivinen CAS-avusteinen (computer algrbra systems) työskentely differentiaalilaskennan aiheiden parissa edistää käsitteellistä oppimista. Tutkimustulokset viittasivat myös siihen, että laskutaidot voivat kestää pidempään, kun ne on rakennettu ymmärtämisen pohjalle (Repo, 1996.) Tästä syystä en nähnyt tarpeelliseksi enää verrata tutkivaa pienryhmätyskentelyä perinteiseen matematiikan opetukseen. Sen sijaan, päätin keskittyä kehittämään uutta opetusmenetelmää erityisesti omassa työssäni. Tulosteni pohjalta on selvää, että työmuoto vaatii uusien sosiaalisten ja sosiomatemaattisten normien neuvottelua. Menetelmä ei aivan noin vain soveltunut perinteiseen luokkahuoneeseen. Tarvitaan lisää kehitystyötä sen suomien mahdollisuuksien hyödyntämiseksi. Toisaalta tutkivaan opiskeluun (inquiry mathematics) liittyvät uudet normit ja niiden neuvottelu ovat mielestäni vakavsti otettava ratkaisuvaihotehto pyrkiessämme syvempiin tavoitteisiin, joita opetussuunnitelmissa mainitaan ja joihin meillä ei ole aikaisemmin ollut välineitä. Tutkimusprosessini paljasti joitain perinteisen koulumatematiikan ei-toivottuja seurauksia. Tässä yhteydessä katson itseni osaksi sitä koulumatematiikan kulttuuria, jonka tutkiva pieneryhmätyöskentely haastaa.

Tutkimukseni pohjalta olen rakentanut kehyksen, jota voi käyttää apuna tutkivaa pienryhmätyöskelyä tukevien sosiaalisten ja sosiomatemaattisten normien neuvottelussa.

Emergentin viitekehyksen sosiaaliset ja sosiomatemaattiset normit sekä oppimismahdollisuudet olivat erittäin hyödyllisiä käsitteitä pyrkiessäni kuvaamaan kokeilukurssini ilmiöitä pintaa syvemmältä. Teoria kuitenkin käsittelee lähinnä vain luokkahuoneen tason ilmiöitä. Aineistossani näkyy, kuinka tyttöjen ja poikien erilaiset sosiolingvistiset alakulttuurit olivat läsnä pienryhmäkeskusteluissa jopa niin, että ne olivat yhteydessä oppimismahdollisuuksien syntymiseen. Jo klassisessa artikkelissaan Cobb ja Yackel (1996) hahmottelevat viitekehyksen laajentamista. He kirjoittavat siitä, kuinka luokkahuoneen mikrokulttuuri on osa kyseisen koulun kulttuuria ja kuinka joskus tarvitaan myös koulutuksen kansallisen kontekstin huomioon ottamista. Myöhemmin Cobb ja Hodge (2002) ovat edelleen hyödyntäneet sosiokulttuurisia teorioita sen kuvaamisessa, kuinka kulttuurinen monimuotoisuus tulee luokkahuoneeseen opiskelijoiden erilaisten identiteettien mukana. Vaikka edellämainitut laajennukset ovatkin inspiroivia, näen kuitekin ongelmalliseksi tavan, jolla Cobb yhdistelee teorioita lähtökohdiltaan hyvin erilaisista traditioista. Mielestäni emergentin viitekehyksen laajentamisesta pitäisi vielä käydä lisää keskustelua.

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